(3) $y'' + y = \sin x$

$y'' + y = 0$

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1}$

$m = \pm i \quad (\alpha = 0, \beta = 1)$

$y_c = C_1 \cos x + C_2 \sin x$

$\boxed{y_1 = \cos x \quad y_2 = \sin x}$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$= \cos^2 x + \sin^2 x$

$= 1$

$f(x) = \sin x$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sin x & \cos x \end{vmatrix}$$

$= -\sin^2 x$

$u_1' = -\sin^2 x$

$u_1 = \int -\sin^2 x \, dx$

$= \int -\frac{1}{2}(1 - \cos 2x)\, dx$

$= -\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sin x \end{vmatrix}$$
$$= \cos x \sin x$$

$$u_2' = \cos x \sin x$$
$$u_2 = \int \cos x \sin x \, dx$$
$$= \int \tfrac{1}{2} \sin 2x \, dx$$
$$= -\tfrac{1}{4} \cos 2x$$

$$y_p = u_1 y_1 + u_2 y_2$$
$$= -\tfrac{1}{2}\left(x - \tfrac{\sin 2x}{2}\right)\cos x \;-\; \tfrac{1}{4}\cos 2x \sin x$$

$$y = y_c + y_p$$

NOTE: Depending on your choice of y_1 and y_2, u_1' and u_2',
and depending on how you integrate
you may get an alternative correct answer for y_p:

$$y_p = -\tfrac{1}{2}\left(x - \tfrac{\sin 2x}{2}\right)\cos x + \left(\tfrac{1}{2}\sin^2 x\right)\sin x$$
Then $y = y_c + y_p$

Note: Coursepack simplifies y_p to $-\frac{1}{2}x\cos x + C_3 \sin x$

$\Rightarrow y = y_c + y_p$

$= C_1 \cos x + C_2 \sin x - \frac{1}{2} x \cos x$

No need to do this.

(5) $y'' + y = \cos^2 x$

$y'' + y = 0$

$\vdots$

$y_c = c_1 \cos x + c_2 \sin x$ (see #3 of Section 4.6)

$y_1 = \cos x \quad y_2 = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \cos^2 x & \cos x \end{vmatrix} = -\sin x \cos^2 x$$

$u_1' = -\sin x \cos^2 x$

$$u_1 = \int -\sin x \cos^2 x \, dx = \int u^2 \, du = \frac{u^3}{3} = \frac{1}{3}\cos^3 x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \cos^2 x \end{vmatrix} = \cos^3 x$$

$u_2' = \cos^3 x$

$$u_2 = \int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx = \int (1-u^2) \, du = u - \frac{u^3}{3} = \sin x - \frac{\sin^3 x}{3}$$

$$y_p = u_1 y_1 + u_2 y_2 = \frac{1}{3}\cos^3 x \cdot \cos x + \left(\sin x - \frac{\sin^3 x}{3}\right)\sin x = \frac{1}{3}\cos^4 x + \sin^2 x - \frac{\sin^4 x}{3}$$

$\rightarrow$

Then $y = y_c + y_p$

Note: Coursepath simplifies y_p to $\frac{1}{2} - \frac{1}{6}\cos 2x$

$\Rightarrow y = y_c + y_p$

$y = C_1 \cos x + C_2 \sin x + \frac{1}{2} - \frac{1}{6}\cos 2x$

No need to do this.

(11) $y'' + 3y' + 2y = \dfrac{1}{1+e^x}$

$$y'' + 3y' + 2y = 0$$
$$m^2 + 3m + 2 = 0$$
$$(m+2)(m+1) = 0$$
$$m = -2, -1$$
$$y_c = C_1 e^{-2x} + C_2 e^{-x}$$
$$y_1 = e^{-2x} \quad y_2 = e^{-x}$$

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}$$
$$= -e^{-3x} + 2e^{-3x}$$
$$= e^{-3x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{1}{1+e^x} & -e^{-x} \end{vmatrix} = \frac{-e^{-x}}{1+e^x}$$

$$W_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \frac{1}{1+e^x} \end{vmatrix} = \frac{e^{-2x}}{1+e^x}$$

$$u_1' = \frac{W_1}{W} = \frac{-e^{2x}}{1+e^x}$$

$$u_2' = \frac{W_2}{W} = \frac{e^x}{1+e^x}$$

$\rightarrow$

$$u_1 = \int \frac{-e^{2x}}{1+e^x}\,dx$$

$$= \int \frac{-u}{1+u}\,du$$

Long Division

$$\begin{array}{r|l} & -1 \\ -u & u+1 \\ & -\ u \\ \hline & 1 \end{array}$$

$$= \int \left[-1 + \frac{1}{1+u}\right] du$$

$$= -u + \ln|1+u|$$

$$= -e^x + \ln|1+e^x|$$

or $-e^x + \ln(1+e^x)$

$$u_2 = \int \frac{e^x}{1+e^x}\,dx$$

$$= \int \frac{du}{u}$$

$$= \ln|u|$$

$$= \ln|1+e^x|$$

or $\ln(1+e^x)$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= e^{-2x}\left[-e^x + \ln(1+e^x)\right] + e^{-x}\ln(1+e^x)$$

$$= -e^{-x} + (e^{-x} + e^{-2x})\ln(1+e^x)$$

$$y = y_c + y_p$$

$$= C_1 e^{-2x} + C_2 e^{-x} - e^{-x} + (e^{-x} + e^{-2x})\ln(1+e^x)$$

or $C_1 e^{-2x} + C_3 e^{-x} + (e^{-x} + e^{-2x})\ln(1+e^x)$

(13) $y'' + 3y' + 2y = \sin e^x$

$y'' + 3y' + 2y = 0$

$m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

$y_c = C_1 e^{-x} + C_2 e^{-2x}$

$y_1 = e^{-x} \quad y_2 = e^{-2x}$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$= -2e^{-3x} + e^{-3x}$

$= -e^{-3x}$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \sin e^x & -2e^{-2x} \end{vmatrix}$$

$= -e^{-2x} \sin e^x$

$u_1' = \dfrac{W_1}{W}$

$= e^x \sin e^x$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \sin e^x \end{vmatrix}$$

$= e^{-x} \sin e^x$

$u_2' = \dfrac{W_2}{W}$

$= -e^{2x} \sin e^x$

$\rightarrow$

$$u_1 = \int e^x \sin e^x \, dx$$
$$= \int \sin u \, du$$
$$= -\cos u$$
$$= -\cos e^x$$

$$u_2 = \int -e^{2x} \sin e^x \, dx$$
$$= \int -e^x \sin e^x \left(e^x \, dx\right)$$
$$= \int -u \sin u \, du$$

	D	I
$\oplus$	$-u$	$\sin u$
$\ominus$	-1	$-\cos u$
		$-\sin u$

$$u_2 = u \cos u - \sin u$$
$$= e^x \cos e^x - \sin e^x$$

$$y_p = u_1 y_1 + u_2 y_2$$
$$= -\cos e^x \left(e^{-x}\right) + \left(e^x \cos e^x - \sin e^x\right) e^{-2x}$$
$$= -e^{-x} \cos e^x + e^{-x} \cos e^x - e^{-2x} \sin e^x$$
$$= -e^{-2x} \sin e^x$$

$$y = y_c + y_p$$
$$y = C_1 e^{-x} + C_2 e^{-2x} - e^{-2x} \sin e^x$$

(15) $y'' + 2y' + y = e^{-t}\ln t$

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, -1$

$y_c = C_1 e^{-t} + C_2 t e^{-t}$

$$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix}$$

$= e^{-t}(-te^{-t} + e^{-t}) + e^{-t}(te^{-t})$

$= e^{-2t}$

$$W_1 = \begin{vmatrix} 0 & te^{-t} \\ e^{-t}\ln t & -te^{-t} + e^{-t} \end{vmatrix} \qquad W_2 = \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & e^{-t}\ln t \end{vmatrix}$$

$= -te^{-2t}\ln t \qquad = e^{-2t}\ln t$

$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$

$= -t\ln t \qquad = \ln t$

$u_1 = \int -t\ln t \, dt \qquad u_2 = \int \ln t \, dt$

$\rightarrow$

$$u_1 = \int -t\ln t\,dt$$

$$\begin{array}{c|c} D & I \\ \hline \ln t & -t \\ \frac{1}{t} & -\frac{t^2}{2} \end{array}$$

$$u_1 = -\frac{t^2}{2}\ln t + \int \frac{t}{2}\,dt$$

$$= -\frac{t^2}{2}\ln t + \frac{t^2}{4}$$

$$u_2 = \int \ln t\,dt$$

$$\begin{array}{c|c} D & I \\ \hline \ln t & 1 \\ \frac{1}{t} & t \end{array}$$

$$u_2 = t\ln t - \int 1\,dt$$

$$= t\ln t - t$$

$$y_p = \left(-\frac{t^2}{2}\ln t + \frac{t^2}{4}\right)e^{-t} + (t\ln t - t)te^{-t}$$

$$= -\frac{t^2}{2}e^{-t}\ln t + \frac{t^2}{4}e^{-t} + t^2e^{-t}\ln t - t^2e^{-t}$$

$$= \frac{t^2}{2}e^{-t}\ln t - \frac{3t^2}{4}e^{-t}$$

$$y = y_c + y_p$$

$$y = C_1e^{-t} + C_2te^{-t} + \frac{t^2}{2}e^{-t}\ln t - \frac{3t^2}{4}e^{-t}$$

(17) $3y'' - 6y' + 6y = e^x \sec x$

$3m^2 - 6m + 6 = 0$

$m^2 - 2m + 2 = 0$

$m = \dfrac{2 \pm \sqrt{4 - 4(1)(2)}}{2}$

$m = \dfrac{2 \pm \sqrt{-4}}{2}$

$m = \dfrac{2 \pm 2i}{2}$

$m = 1 \pm i \quad (\alpha = 1, \beta = 1)$

$y_c = e^x (C_1 \cos x + C_2 \sin x)$

$y_1 = e^x \cos x, \quad y_2 = e^x \sin x$

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ -e^x \sin x + e^x \cos x & e^x \cos x + e^x \sin x \end{vmatrix}$$

$= e^x \cos x (e^x \cos x + e^x \sin x) - e^x \sin x (-e^x \sin x + e^x \cos x)$

$= e^{2x} \cos^2 x + e^{2x} \cos x \sin x + e^{2x} \sin^2 x - e^{2x} \cos x \sin x$

$= e^{2x} (\cos^2 x + \sin^2 x)$

$= e^{2x}$

Standard form is

$$y'' - 2y' + 2y = \frac{e^x \sec x}{3}$$

$\boxed{\text{this is } f(x)}$

$$W_1 = \begin{vmatrix} 0 & e^x \sin x \\ \frac{e^x \sec x}{3} & e^x \cos x + e^x \sin x \end{vmatrix}$$

$$= \frac{-e^{2x} \sin x \sec x}{3}$$

$$= \frac{-e^{2x} \tan x}{3}$$

$$u_1' = \frac{W_1}{W}$$

$$= -\frac{\tan x}{3}$$

$$u_1 = \int -\frac{\tan x}{3}\, dx$$

$$= \frac{1}{3} \ln|\cos x|$$

$$W_2 = \begin{vmatrix} e^x \cos x & 0 \\ -e^x \sin x + e^x \cos x & \frac{e^x \sec x}{3} \end{vmatrix}$$

$$= \frac{e^{2x} \cos x \sec x}{3}$$

$$= \frac{e^{2x}}{3}$$

$$u_2' = \frac{W_2}{W}$$

$$= \frac{1}{3}$$

$$u_2 = \int \frac{1}{3}\, dx$$

$$= \frac{x}{3}$$

$$y_p = \frac{1}{3} \ln|\cos x| e^x \cos x + \frac{x}{3} e^x \sin x$$

$$= \frac{1}{3} e^x \cos x \ln|\cos x| + \frac{1}{3} x e^x \sin x$$

OR $$-\frac{1}{3} e^x \cos x \ln|\sec x| + \frac{1}{3} x e^x \sin x$$

$$y = y_c + y_p$$

(21) $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}, \; y(0) = 1, \; y'(0) = 0$

$m^2 + 2m - 8 = 0$

$(m+4)(m-2) = 0$

$m = -4, 2$

$y_c = C_1 e^{-4x} + C_2 e^{2x}$

$y_1 = e^{-4x}, \quad y_2 = e^{2x}$

$$W = \begin{vmatrix} e^{-4x} & e^{2x} \\ -4e^{-4x} & 2e^{2x} \end{vmatrix}$$

$= 2e^{-2x} + 4e^{-2x}$

$= 6e^{-2x}$

$$W_1 = \begin{vmatrix} 0 & e^{2x} \\ 2e^{-2x} - e^{-x} & 2e^{2x} \end{vmatrix}$$

$= -e^{2x}(2e^{-2x} - e^{-x})$

$= -2 + e^{x}$

$u_1' = \frac{W_1}{W}$

$= \frac{-2 + e^{x}}{6e^{-2x}}$

$= e^{2x}\left(\frac{-1}{3} + \frac{e^{x}}{6}\right)$

$= -\frac{e^{2x}}{3} + \frac{e^{3x}}{6}$

$$W_2 = \begin{vmatrix} e^{-4x} & 0 \\ -4e^{-4x} & 2e^{-2x} - e^{-x} \end{vmatrix}$$

$= 2e^{-6x} - e^{-5x}$

$u_2' = \frac{W_2}{W}$

$= \frac{2e^{-6x} - e^{-5x}}{6e^{-2x}}$

$= e^{2x}\left(\frac{e^{-6x}}{3} - \frac{e^{-5x}}{6}\right)$

$= \frac{e^{-4x}}{3} - \frac{e^{-3x}}{6}$

$\longrightarrow$

$$u_1' = -\frac{e^{2x}}{3} + \frac{e^{3x}}{6} \qquad u_2' = \frac{e^{-4x}}{3} - \frac{e^{-3x}}{6}$$

$$u_1 = -\frac{e^{2x}}{6} + \frac{e^{3x}}{18} \qquad u_2 = -\frac{e^{-4x}}{12} + \frac{e^{-3x}}{18}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = \left(-\frac{e^{2x}}{6} + \frac{e^{3x}}{18}\right)e^{-4x} + \left(-\frac{e^{-4x}}{12} + \frac{e^{-3x}}{18}\right)e^{2x}$$

$$= -\frac{e^{-2x}}{6} + \frac{e^{-x}}{18} - \frac{e^{-2x}}{12} + \frac{e^{-x}}{18}$$

$$= -\frac{e^{-2x}}{4} + \frac{e^{-x}}{9}$$

$$y = y_c + y_p$$

$$\boxed{y = C_1 e^{-4x} + C_2 e^{2x} - \frac{e^{-2x}}{4} + \frac{e^{-x}}{9}}$$

$y=1$, $x=0$: $\quad 1 = C_1 + C_2 - \frac{1}{4} + \frac{1}{9}$

$$C_1 + C_2 = \frac{41}{36} \qquad (1)$$

$$y' = -4C_1 e^{-4x} + 2C_2 e^{2x} + \frac{e^{-2x}}{2} - \frac{e^{-x}}{9}$$

$y'=0$, $x=0$: $\quad 0 = -4C_1 + 2C_2 + \frac{1}{2} - \frac{1}{9}$

$$-4C_1 + 2C_2 = -\frac{7}{18} \qquad (2)$$

$\rightarrow$

4×1 : $4C_1 + 4C_2 = \frac{164}{36}$

2 : $-4C_1 + 2C_2 = \frac{-14}{36}$

$+$

$6C_2 = \frac{150}{36}$

$C_2 = \frac{25}{36}$

$C_2 = \frac{25}{36} \rightarrow 1$: $C_1 + \frac{25}{36} = \frac{41}{36}$

$C_1 = \frac{16}{36}$ or $\frac{4}{9}$

$$y = C_1 e^{-4x} + C_2 e^{2x} - \frac{e^{-2x}}{4} + \frac{e^{-x}}{9}$$

$$y = \frac{4}{9} e^{-4x} + \frac{25}{36} e^{2x} - \frac{e^{-2x}}{4} + \frac{e^{-x}}{9}$$