Section 4.4

(5) $\frac{1}{4}y'' + y' + y = x^2 - 2x$

$$\frac{1}{4}m^2 + m + 1 = 0$$

$$\left(\frac{1}{2}m + 1\right)^2 = 0$$

$$m = -2, -2$$

$$y_c = C_1 e^{-2x} + C_2 x e^{-2x}$$

$$\begin{cases} y_p = Ax^2 + Bx + C \\ y_p' = 2Ax + B \\ y_p'' = 2A \end{cases}$$

$y_p \to DE$: $\frac{A}{2} + 2Ax + B + Ax^2 + Bx + C = x^2 - 2x$

$$Ax^2 + (2A + B)x + \left(\frac{A}{2} + B + C\right) = x^2 - 2x$$

$$\boxed{A = 1}$$

$$\boxed{\begin{aligned} 2A + B &= -2 \\ 2 + B &= -2 \\ B &= -4 \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{A}{2} + B + C &= 0 \\ \frac{1}{2} - 4 + C &= 0 \\ -\frac{7}{2} &= -C \\ C &= \frac{7}{2} \end{aligned}}$$

$$y_p = x^2 - 4x + \frac{7}{2}$$

$$y = C_1 e^{-2x} + C_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}$$

(7) $y'' + 3y = -48x^2 e^{3x}$

$m^2 + 3 = 0$

$m^2 = -3$

$m = \pm\sqrt{-3}$

$m = \pm\sqrt{3}\, i$

$\alpha = 0 \quad \beta = \sqrt{3}$

$y_c = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$y_c = C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x$

$y_p = (Ax^2 + Bx + C)e^{3x}$

$y_p' = (Ax^2 + Bx + C)3e^{3x} + (2Ax + B)e^{3x}$

$= (3Ax^2 + 3Bx + 2Ax + 3C + B)e^{3x}$

$y_p'' = (3Ax^2 + 3Bx + 2Ax + 3C + B)3e^{3x} + (6Ax + 3B + 2A)e^{3x}$

$= (9Ax^2 + 9Bx + 6Ax + 9C + 3B + 6Ax + 3B + 2A)e^{3x}$

$= (9Ax^2 + 9Bx + 12Ax + 9C + 6B + 2A)e^{3x}$

$y_p \to DE$: $(9Ax^2 + 9Bx + 12Ax + 9C + 6B + 2A)e^{3x} + 3(Ax^2 + Bx + C)e^{3x}$

$= -48x^2 e^{3x}$

$12A = -48 \Rightarrow A = -4$

$9B + 12A + 3B = 0 \Rightarrow 12B - 48 = 0 \Rightarrow B = 4$

$9C + 6B + 2A + 3C = 0 \Rightarrow 12C + 24 - 8 = 0 \Rightarrow C = -\frac{4}{3}$

$y = y_c + y_p$

$y = C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x + (-4x^2 + 4x - \frac{4}{3})e^{3x}$

(9) $y'' - y' = -3$

$m^2 - m = 0$

$m(m-1) = 0$

$m = 0, 1$

$y_c = C_1 + C_2 e^x$

~~$y_p = A$~~ Bad Case

$$\begin{cases} y_p = Ax \\ y_p' = A \\ y_p'' = 0 \end{cases}$$

$y_p \to DE$: $0 - A = -3$

$A = 3$

$y = y_c + y_p$

$y = C_1 + C_2 e^x + 3x$

(13) $y'' + 4y = 3\sin 2x$

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm\sqrt{-4}$$

$$m = \pm 2i \qquad (\alpha = 0,\ \beta = 2)$$

$$y_c = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$$

$$y_c = C_1\cos 2x + C_2 \sin 2x$$

$\cancel{y_p = A\cos 2x + B\sin 2x}$ Bad Case

$$\begin{cases} y_p = Ax\cos 2x + Bx\sin 2x \\ y_p' = -2Ax\sin 2x + A\cos 2x + 2Bx\cos 2x + B\sin 2x \\ y_p'' = -4Ax\cos 2x - 2A\sin 2x - 2A\sin 2x + 2B\cos 2x + 2B\cos 2x \\ \qquad -4Bx\sin 2x \end{cases}$$

$y_p \to DE$: $$-\underline{4Ax\cos 2x} - 4A\sin 2x - \underline{4Bx\sin 2x} + 4B\cos 2x + \underline{4Ax\cos 2x} + \underline{4Bx\sin 2x} = 3\sin 2x$$

$$-4A = 3 \implies A = -3/4$$

$$4B = 0 \implies B = 0$$

$$y_p = -\frac{3}{4}x\cos 2x$$

$$y = y_c + y_p$$

$$y = C_1\cos 2x + C_2\sin 2x - \frac{3}{4}x\cos 2x$$

(19) $y'' + 2y' + y = \sin x + 3\cos 2x$

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, -1$

$y_c = C_1 e^{-x} + C_2 x e^{-x}$

$$\begin{cases} y_p = A\cos x + B\sin x + C\cos 2x + D\sin 2x \\ y_p' = -A\sin x + B\cos x - 2C\sin 2x + 2D\cos 2x \\ y_p'' = -A\cos x - B\sin x - 4C\cos 2x - 4D\sin 2x \end{cases}$$

$y_p \to DE$:

$$\begin{aligned} &-A\cos x - B\sin x - 4C\cos 2x - 4D\sin 2x \\ &-2A\sin x + 2B\cos x - 4C\sin 2x + 4D\cos 2x \\ &+A\cos x + B\sin x + C\cos 2x + D\sin 2x \\ &= \sin x + 3\cos 2x \end{aligned}$$

sin x term: $-B - 2A + B = 1 \Rightarrow A = -\frac{1}{2}$

cos x term: $-A + 2B + A = 0 \Rightarrow B = 0$

sin2x term: $-4D - 4C + D = 0 \Rightarrow -4C - 3D = 0$ (1)

cos2x term: $-4C + 4D + C = 3$ $\quad -3C + 4D = 3$ (2)

$3 \times (1)$: $-12C - 9D = 0$

$-4 \times (2)$ $\quad +$ $12C - 16D = -12$

$-25D = -12$

$D = \frac{12}{25}$

$D = \frac{12}{25} \to (1)$: $-4C - \frac{36}{25} = 0 \Rightarrow C = \frac{-9}{25}$

$y_p = -\frac{1}{2}\cos x - \frac{9}{25}\cos 2x + \frac{12}{25}\sin 2x$

$y = C_1 e^{-x} + C_2 x e^{-x} - \frac{1}{2}\cos x - \frac{9}{25}\cos 2x + \frac{12}{25}\sin 2x$

(21) $y''' - 6y'' = 3 - \cos x$

$m^3 - 6m^2 = 0$

$m^2(m-6) = 0$

$m = 0, 0, 6$

$y_c = C_1 + C_2 x + C_3 e^{6x}$

$\cancel{y_p = A} + B\cos x + C\sin x$ Bad Case

$$\begin{cases} y_p = Ax^2 + B\cos x + C\sin x \\ y_p' = 2Ax - B\sin x + C\cos x \\ y_p'' = 2A - B\cos x - C\sin x \\ y_p''' = 0 + B\sin x - C\cos x \end{cases}$$

$y_p \to DE$: $B\sin x - C\cos x - 12A + 6B\cos x + 6C\sin x = 3 - \cos x$

$-12A = 3 \Rightarrow A = -1/4$

Sin x term: $B + 6C = 0$ (1)

Cos x term: $-C + 6B = -1$ (2)

$-6\times(1)$: $-6B - 36C = 0$

(2): $+\quad 6B - C = -1$

$-37C = -1$

$C = \frac{1}{37}$

$C = \frac{1}{37} \to (1)$: $B + \frac{6}{37} = 0 \Rightarrow B = \frac{-6}{37}$

$y_p = -\frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x$

$y = C_1 + C_2 x + C_3 e^{6x} - \frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x$

(27) $y'' + 4y = -2, \quad y\left(\frac{\pi}{8}\right) = \frac{1}{2}, \quad y'\left(\frac{\pi}{8}\right) = 2$

$$m^2 + 4 = 0$$
$$m^2 = -4$$
$$m = \pm\sqrt{-4}$$
$$m = \pm 2i \qquad (\alpha = 0, \beta = 2)$$

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

$$\begin{cases} y_p = A \\ y_p' = 0 \\ y_p'' = 0 \end{cases}$$

$y_p \to DE$:
$$4A = -2$$
$$A = \frac{-1}{2}$$

$$y_p = \frac{-1}{2}$$

$$\boxed{y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{2}}$$

$$\boxed{\begin{array}{l} \text{Note: } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \\ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \end{array}}$$

$y = \frac{1}{2}$, $x = \frac{\pi}{8}$:
$$\frac{1}{2} = C_1\left(\frac{1}{\sqrt{2}}\right) + C_2\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2}$$
$$C_1\left(\frac{1}{\sqrt{2}}\right) + C_2\left(\frac{1}{\sqrt{2}}\right) = 1$$
$$C_1 + C_2 = \sqrt{2} \qquad (1)$$

$$y' = -2C_1 \sin 2x + 2C_2 \cos 2x$$

$y' = 2$, $x = \frac{\pi}{8}$:
$$2 = -2C_1\left(\frac{1}{\sqrt{2}}\right) + 2C_2\left(\frac{1}{\sqrt{2}}\right)$$
$$-2C_1 + 2C_2 = 2\sqrt{2} \qquad (2)$$

$\rightarrow$

27 Cont'd

$$2 \times 1: \quad 2C_1 + 2C_2 = 2\sqrt{2}$$
$$2: \quad + \; -2C_1 + 2C_2 = 2\sqrt{2}$$
$$4C_2 = 4\sqrt{2}$$
$$C_2 = \sqrt{2}$$

$C_2 = \sqrt{2} \rightarrow 1$: $\quad C_1 + \sqrt{2} = \sqrt{2}$

$$C_1 = 0$$

$$y = \sqrt{2} \sin 2x - \frac{1}{2}$$

(29) $5y'' + y' = -6x, \quad y(0)=0, \quad y'(0) = -10$

$5m^2 + m = 0$

$m(5m+1) = 0$

$m = 0, \frac{-1}{5}$

$y_c = C_1 + C_2 e^{-x/5}$

$\left\{ \begin{array}{l} \cancel{y_p = Ax+B} \quad \text{Bad Case} \quad y_p = Ax^2 + Bx \\ y_p' = 2Ax + B \\ y_p'' = 2A \end{array} \right.$

$y_p \to DE: \quad 10A + 2Ax + B = -6x$

$2A = -6 \Rightarrow A = -3$

$10A + B = 0 \Rightarrow -30 + B = 0 \Rightarrow B = 30$

$y_p = -3x^2 + 30x$

$$\boxed{y = C_1 + C_2 e^{-x/5} - 3x^2 + 30x}$$

$\begin{array}{l} y=0 \\ x=0 \end{array}$: $\quad 0 = C_1 + C_2 \quad$ (1)

$y' = -\frac{1}{5} C_2 e^{-x/5} - 6x + 30$

$\begin{array}{l} y'=-10 \\ x=0 \end{array}$: $\quad -10 = -\frac{1}{5} C_2 + 30 \quad$ (2)

$-40 = -\frac{1}{5} C_2 \quad \rightarrow$

(29) Cont'd

$$C_2 = 200$$

$C_2 = 200 \rightarrow (1)$: $C_1 + 200 = 0$

$$C_1 = -200$$

$$y = -200 + 200e^{-x/5} - 3x^2 + 30x$$

(31) $y'' + 4y' + 5y = 35e^{-4x}$, $y(0) = -3$, $y'(0) = 1$

$m^2 + 4m + 5 = 0$

$m = \dfrac{-4 \pm \sqrt{16 - 4(1)(5)}}{2}$

$m = \dfrac{-4 \pm \sqrt{-4}}{2}$

$m = \dfrac{-4 \pm 2i}{2}$

$m = -2 \pm i$

$y_c = e^{-2x}(C_1 \cos x + C_2 \sin x)$

$$\begin{cases} y_p = Ae^{-4x} \\ y_p' = -4Ae^{-4x} \\ y_p'' = 16Ae^{-4x} \end{cases}$$

$y_p \to DE$: $16Ae^{-4x} - 16Ae^{-4x} + 5Ae^{-4x} = 35e^{-4x}$

$5Ae^{-4x} = 35e^{-4x}$

$A = 7$

$y = y_c + y_p$

$\boxed{y = e^{-2x}(C_1 \cos x + C_2 \sin x) + 7e^{-4x}}$

$\begin{matrix} y = -3 \\ x = 0 \end{matrix}$: $-3 = C_1 + 7$

$C_1 = -10$

$\boxed{y = e^{-2x}(-10 \cos x + C_2 \sin x) + 7e^{-4x}}$ →

$$y' = e^{-2x}(10\sin x + C_2\cos x) - 2e^{-2x}(-10\cos x + C_2\sin x) - 28e^{-4x}$$

$y'=1$, $x=0$:

$$1 = C_2 - 2(-10) - 28$$

$$C_2 = 9$$

$$y = e^{-2x}(-10\cos x + 9\sin x) + 7e^{-4x}$$