Section 4.3

(3) $y'' - y' - 6y = 0$

$m^2 - m - 6 = 0$

$(m-3)(m+2) = 0$

$m = -2, 3$

$y = C_1 e^{-2x} + C_2 e^{3x}$

or $y = C_1 e^{3x} + C_2 e^{-2x}$

(5) $y'' + 8y' + 16y = 0$

$m^2 + 8m + 16 = 0$

$(m+4)^2 = 0$

$m = -4, -4$

$y = C_1 e^{-4x} + C_2 x e^{-4x}$

(7) $12y'' - 5y' - 2y = 0$

$12m^2 - 5m - 2 = 0$

$$m = \frac{5 \pm \sqrt{25 - 4(12)(-2)}}{24}$$

$$m = \frac{5 \pm \sqrt{121}}{24}$$

$$m = \frac{5 \pm 11}{24}$$

$m = \frac{2}{3}, \frac{-1}{4}$

$y = C_1 e^{\frac{2x}{3}} + C_2 e^{-x/4}$

(9) $y'' + 9y = 0$

$m^2 + 9 = 0$

$m^2 = -9$

$m = \pm\sqrt{-9}$

$m = \pm 3i$

$\alpha = 0, \beta = 3$

$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$y = C_1 \cos 3x + C_2 \sin 3x$

(11) $y'' - 4y' + 5y = 0$

$m^2 - 4m + 5 = 0$

$$m = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2}$$

$$m = \frac{4 \pm \sqrt{-4}}{2}$$

$$m = \frac{4 \pm 2i}{2}$$

$$m = 2 \pm i$$

$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$y = e^{2x}(C_1 \cos x + C_2 \sin x)$

(13) $3y'' + 2y' + y = 0$

$3m^2 + 2m + 1 = 0$

$$m = \frac{-2 \pm \sqrt{4 - 4(3)(1)}}{6}$$

$$m = \frac{-2 \pm \sqrt{-8}}{6} \leftarrow \sqrt{4}\sqrt{2}\sqrt{-1}$$

$$m = \frac{-2 \pm 2\sqrt{2}i}{6}$$

$$m = \frac{-1}{3} \pm \frac{\sqrt{2}}{3}i$$

$\rightarrow$

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

$$y = e^{-x/3}\left(C_1 \cos \frac{\sqrt{2}\,x}{3} + C_2 \sin \frac{\sqrt{2}\,x}{3}\right)$$

(15)

$$y''' - 4y'' - 5y' = 0$$

$$m^3 - 4m^2 - 5m = 0$$

$$m(m^2 - 4m - 5) = 0$$

$$m(m-5)(m+1) = 0$$

$$m = 0, 5, -1$$

$$y = C_1 + C_2 e^{5x} + C_3 e^{-x}$$

or $y = C_1 + C_2 e^{-x} + C_3 e^{5x}$ etc.

(17)

$$y''' - 5y'' + 3y' + 9y = 0$$

$$m^3 - 5m^2 + 3m + 9 = 0$$

By Rational Roots Theorem,

Possibilities for p: $\pm 1, \pm 3, \pm 9$

" q: ± 1

Possibilities for $\frac{p}{q}$: $\underbrace{\pm 1, \pm 3, \pm 9}_{\text{possible roots}}$

$m=-1 \rightarrow m^3-5m^2+3m+9=0$ ✓

$m=-1$ is a root of the auxiliary equation

$\Rightarrow (m+1)$ is a factor "

$$
\begin{array}{r}
m^2-6m+9 \\
(m+1)\overline{\big)\, m^3-5m^2+3m+9} \\
-(m^3+m^2) \\
\hline
-6m^2+3m+9 \\
-(-6m^2-6m) \\
\hline
9m+9 \\
-(9m+9) \\
\hline
0
\end{array}
$$

Auxiliary equation: $(m+1)(m^2-6m+9)=0$

$(m+1)(m-3)^2=0$

$m=-1, 3, 3$

$$y = C_1 e^{-x} + C_2 e^{3x} + C_3 x e^{3x}$$

(21) $y''' + 3y'' + 3y' + y = 0$

$m^3 + 3m^2 + 3m + 1 = 0$

Possibilities for p: ± 1

" q: ± 1

" $\frac{p}{q}$: ± 1 ← possible roots

$m = -1 \rightarrow m^3 + 3m^2 + 3m + 1 = 0$ ✓

$m = -1$ is a root of auxiliary equation

$m+1$ is a factor "

$$
\begin{array}{r}
m^2 + 2m + 1 \\
(m+1)\ \overline{)\ m^3 + 3m^2 + 3m + 1} \\
-(m^3 + m^2) \qquad\qquad \\
\hline
2m^2 + 3m + 1 \\
-(2m^2 + 2m) \qquad \\
\hline
m + 1 \\
-(m+1) \\
\hline
0
\end{array}
$$

$m^3 + 3m^2 + 3m + 1 = 0$

$(m+1)(m^2 + 2m + 1) = 0$

$(m+1)(m+1)^2 = 0$

$(m+1)^3 = 0$

$m = -1, -1, -1$

$y = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x}$

(23) $y^{(4)} + y''' + y'' = 0$

$m^4 + m^3 + m^2 = 0$

$m^2(m^2+m+1) = 0$

$m = 0, 0$

$m = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2}$

$m = \frac{-1 \pm \sqrt{-3}}{2}$

$m = \frac{-1 \pm \sqrt{3}\, i}{2}$

$\alpha = -\frac{1}{2} \quad \beta = \frac{\sqrt{3}}{2}$

$y = C_1 e^{0x} + C_2 x e^{0x} + e^{-\frac{x}{2}}\left(C_3 \cos \frac{\sqrt{3}}{2}x + C_4 \sin \frac{\sqrt{3}}{2}x\right)$

$y = C_1 + C_2 x + e^{-\frac{x}{2}}\left(C_3 \cos \frac{\sqrt{3}}{2}x + C_4 \sin \frac{\sqrt{3}}{2}x\right)$

(29) $y'' + 16y = 0, \quad y(0) = 2, \quad y'(0) = -2$

$m^2 + 16 = 0$

$m^2 = -16$

$m = \pm\sqrt{-16}$

$m = \pm 4i$

$y = e^{0x}(C_1 \cos 4x + C_2 \sin 4x)$

$\boxed{y = C_1 \cos 4x + C_2 \sin 4x}$ $\rightarrow$

$\begin{matrix} y=2 \\ x=0 \end{matrix}$: $2 = C_1 + 0$

$C_1 = 2$

$$\boxed{y = 2\cos 4x + C_2 \sin 4x}$$

$y' = -8\sin 4x + 4C_2 \cos 4x$

$\begin{matrix} y'=-2 \\ x=0 \end{matrix}$: $-2 = 0 + 4C_2$

$C_2 = \frac{-1}{2}$

$$y = 2\cos 4x - \frac{1}{2}\sin 4x$$

(31) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 5y = 0$, $y(1)=0$, $y'(1)=2$

$m^2 - 4m - 5 = 0$

$(m-5)(m+1) = 0$

$m = -1, 5$

$$\boxed{y = C_1 e^{-t} + C_2 e^{5t}}$$

$\begin{matrix} y=0 \\ t=1 \end{matrix}$: $0 = C_1 e^{-1} + C_2 e^{5}$ (1)

$$\boxed{y' = -C_1 e^{-t} + 5C_2 e^{5t}}$$

$\begin{matrix} y'=2 \\ t=1 \end{matrix}$: $2 = -C_1 e^{-1} + 5C_2 e^{5}$ (2)

$\rightarrow$

① + ② : $2 = 6C_2e^5$

$$C_2 = \frac{2}{6e^5} = \frac{1}{3}e^{-5}$$

$C_2 = \frac{1}{3}e^{-5} \to$ ① : $0 = C_1e^{-1} + \left(\frac{1}{3}e^{-5}\right)e^5$

$$0 = C_1e^{-1} + \frac{1}{3}$$

$$\frac{-1}{3} = C_1e^{-1}$$

$$-\frac{1}{3}e = C_1$$

$$y = C_1e^{-t} + C_2e^{5t}$$

$$y = \frac{-1}{3}e \cdot e^{-t} + \frac{1}{3}e^{-5} \cdot e^{5t}$$

$$y = \frac{-1}{3}e^{1-t} + \frac{1}{3}e^{5t-5}$$

or $y = \frac{-1}{3}e^{-(t-1)} + \frac{1}{3}e^{5(t-1)}$

(35) $y''' + 12y'' + 36y' = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -7$

$m^3 + 12m^2 + 36m = 0$

$m(m^2 + 12m + 36) = 0$

$m(m+6)^2 = 0$

$m = 0, -6, -6$

$$\boxed{y = C_1 + C_2 e^{-6x} + C_3 x e^{-6x}}$$

$y=0, x=0: \quad 0 = C_1 + C_2$ (1)

$$\boxed{y' = -6C_2 e^{-6x} + C_3(-6x e^{-6x} + e^{-6x})}$$

$y'=1, x=0: \quad 1 = -6C_2 + C_3$ (2)

$y'' = 36C_2 e^{-6x} + C_3(36x e^{-6x} - 6e^{-6x} - 6e^{-6x})$

$y''=-7, x=0: \quad -7 = 36C_2 + C_3(-12)$ (3)

Eliminate C_2: $6 \times$ (2) + (3)

$6 = -36C_2 + 6C_3$

$-7 = 36C_2 - 12C_3$

$-1 = -6C_3$

$\rightarrow$

$$C_3 = \frac{1}{6}$$

$C_3 = \frac{1}{6} \rightarrow (2)$:

$$1 = -6C_2 + C_3$$
$$1 = -6C_2 + \frac{1}{6}$$
$$\frac{5}{6} = -6C_2$$
$$C_2 = \frac{-5}{36}$$

$C_2 = \frac{-5}{36} \rightarrow (1)$:

$$C_1 + C_2 = 0$$
$$C_1 = -C_2$$
$$C_1 = \frac{5}{36}$$

$$y = C_1 + C_2 e^{-6x} + C_3 x e^{-6x}$$
$$y = \frac{5}{36} - \frac{5}{36} e^{-6x} + \frac{1}{6} x e^{-6x}$$