Section 4.2

(3) $y'' + 16y = 0$, $y_1 = \cos 4x$

$p(x) = 0$

$-\int p(x)\,dx = -\int 0\,dx$

$= -0$

$= 0$

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}\,dx$$

$$= \cos 4x \int \frac{e^0}{\cos^2 4x}\,dx$$

$$= \cos 4x \int \sec^2 4x\,dx$$

$$= \cos 4x \left(\frac{1}{4}\tan 4x\right)$$

$$= \frac{1}{4}\sin 4x$$

Can scale y_2 by any nonzero constant

$y_2 = \sin 4x$

(7) $9y'' - 12y' + 4y = 0, \quad y_1 = e^{2x/3}$

$$y'' - \frac{12}{9}y' + \frac{4}{9}y = 0$$

$$y'' - \frac{4}{3}y' + \frac{4}{9}y = 0$$

$$P(x) = -\frac{4}{3}$$

$$-\int P(x)dx = -\int\left(-\frac{4}{3}\right)dx$$

$$= \frac{4}{3}x$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$= e^{2x/3} \int \frac{e^{4x/3}}{e^{4x/3}} dx$$

$$= e^{2x/3} \int 1 \, dx$$

$$= x e^{2x/3}$$

(9) $x^2y'' - 7xy' + 16y = 0, \quad y_1 = x^4$

$$y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0$$

$$P(x) = -\frac{7}{x}$$

$$-\int P(x)\,dx = \int \frac{7}{x}\,dx$$

$$= 7\ln|x|$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx$$

$$= x^4 \int \frac{e^{7\ln|x|}}{x^8}\,dx$$

$$= x^4 \int \frac{e^{\ln|x|^7}}{x^8}\,dx$$

$$= x^4 \int \frac{|x|^7}{x^8}\,dx$$

$$= x^4 \int \frac{x^7}{x^8}\,dx \qquad (\text{interval: } x > 0)$$

$$= x^4 \int \frac{1}{x}\,dx$$

$$= x^4 \ln|x|$$

(11) $xy'' + y' = 0, \qquad y_1 = \ln x$

$$y'' + \frac{1}{x}y' = 0$$

$$P(x) = \frac{1}{x}$$

$$-\int P(x)dx = -\int \frac{1}{x}dx$$

$$= -\ln|x|$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$= (\ln x) \int \frac{e^{-\ln|x|}}{(\ln x)^2} dx$$

$$= (\ln x) \int \frac{e^{\ln|x|^{-1}}}{(\ln x)^2} dx$$

$$= (\ln x) \int \frac{|x|^{-1}}{(\ln x)^2} dx$$

$$= (\ln x) \int \frac{x^{-1}}{(\ln x)^2} dx \qquad \text{(Interval: } x > 0\text{)}$$

$$\boxed{\begin{aligned} \text{Let } u &= \ln x \\ du &= \frac{1}{x}dx \end{aligned}}$$

$$= (\ln x) \int \frac{du}{u^2}$$

$$= (\ln x) \quad (-u^{-1})$$

$\longrightarrow$

$$= \ln x \, (-1)(\ln x)^{-1}$$

$$= -1$$

Can multiply y_2 by any nonzero constant:

$$y_2 = 1$$

(13) $x^2 y'' - x y' + 2y = 0$, $y_1 = x \sin(\ln x)$

$$y'' - \frac{1}{x} y' + \frac{2}{x^2} y = 0$$

$$P(x) = \frac{-1}{x}$$

$$-\int P(x)\,dx = \int \frac{1}{x}\,dx$$

$$= \ln|x|$$

$$= \ln x \qquad (x > 0)$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$= x \sin(\ln x) \int \frac{e^{\ln x}}{x^2 \sin^2(\ln x)}\,dx$$

$$= x \sin(\ln x) \int \frac{x}{x^2 \sin^2(\ln x)}\,dx$$

→

$$= x \sin(\ln x) \int \frac{\csc^2(\ln x)}{x}\,dx$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{dx}{x} \end{aligned}}$$

$$= x \sin(\ln x) \int \csc^2 u\,du$$

$$= x \sin(\ln x)\,[-\cot u]$$

$$= -x \sin(\ln x)\cot(\ln x)$$

$$= -x\cos(\ln x)$$

Can multiply y_2 by any nonzero constant:

$$y_2 = x\cos(\ln x)$$

(15) $(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0, \quad y_1 = x+1$

$$y'' + \frac{2(1+x)}{1-2x-x^2}\,y' - \frac{2}{1-2x-x^2}\,y = 0$$

$$-\int P(x)\,dx = -\int \frac{2(1+x)}{1-2x-x^2}\,dx$$

$$\boxed{\begin{aligned} u &= 1-2x-x^2 \\ du &= (-2-2x)\,dx \\ &= -2(1+x)\,dx \end{aligned}}$$

$$= \int \frac{du}{u}$$

$\rightarrow$

15 Cont'd

$$= \ln|u|$$
$$= \ln|1-2x-x^2|$$
$$= \ln(1-2x-x^2)$$

Interval:
$1-2x-x^2 > 0$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\, dx$$

$$= (x+1) \int \frac{e^{\ln(1-2x-x^2)}}{(x+1)^2}\, dx$$

$$= (x+1) \int \frac{(1-2x-x^2)}{x^2+2x+1}\, dx$$

Long Division

$$\begin{array}{r|l} & \quad -1 \\ \hline x^2+2x+1 & -x^2-2x+1 \\ & -(-x^2-2x-1) \\ \hline & \qquad\qquad 2 \end{array}$$

$$=(x+1) \int \left[-1 + \frac{2}{x^2+2x+1}\right] dx$$
$$= (x+1) \int \left[-1 + \frac{2}{(x+1)^2}\right] dx$$

(15) Cont'd Recopying:

$$y_2 = (x+1)\int \left[-1 + \frac{2}{(x+1)^2}\right]dx$$

$u = x+1$
$du = dx$
$$\int \frac{2}{(x+1)^2}dx = \int \frac{2\,du}{u^2}$$
$$= -2u^{-1} + C$$
$$= \frac{-2}{x+1} + C$$

$$y_2 = (x+1)\left[-x \quad \frac{-2}{x+1}\right]$$

$$= -x(x+1) \quad -2$$

$$= -x^2 - x - 2$$

Can multiply y_2 by any nonzero constant:

$$y_2 = x^2 + x + 2$$