(3) Let $P =$ population
$t =$ time (years)

$\frac{dP}{dt} = kP$ $\quad P(0) = 500$ $\quad P(10) = 500(1.15)$
$P(10) = 575$

Find $P(30)$ and $\left.\frac{dP}{dt}\right|_{t=30}$

$$\frac{dP}{dt} = kP$$
$$\frac{dP}{P} = k\,dt$$
$$\int \frac{dP}{P} = \int k\,dt$$
$$\ln|P| = kt + C_1$$
$$|P| = e^{kt + C_1}$$
$$|P| = e^{C_1} \cdot e^{kt}$$
$$P = \pm e^{C_1} \cdot e^{kt}$$
$$P = Ce^{kt}$$

$P = 500$, $t = 0$: $\quad 500 = C$

$$P = 500e^{kt}$$

$P = 575$, $t = 10$: $\quad 575 = 500\,e^{10k}$

$\rightarrow$

$$1.15 = e^{10k}$$

$$\ln 1.15 = 10k$$

$$k = 0.1 \ln 1.15$$

$$\boxed{P = 500\, e^{(0.1 \ln 1.15)t}}$$

$P(30) \approx 760$ people

$$\frac{dP}{dt} = 500\,(0.1 \ln 1.15)\, e^{(0.1 \ln 1.15)t}$$

$\left.\frac{dP}{dt}\right|_{30} \approx 11$ people/year

(11) "The info on page 86" refers to the fact that the half-life of C-14 is 5600 years. (As mentioned in class).

Let A = mass of C-14 (g)
t = time (years)

$A(5600) = 0.5A_0$ $\qquad A(0) = A_0$

Find t such that $\qquad A(t) = 0.145A_0$

[because 85.5% decayed ⇒ 14.5% remaining]

$$\frac{dA}{dt} = kA$$

$$\frac{dA}{A} = k\,dt$$

$$\int \frac{dA}{A} = \int k\,dt$$

$$\ln|A| = kt + C_1$$

$$|A| = e^{kt+C_1}$$

$$|A| = e^{C_1} \cdot e^{kt}$$

$$A = \pm e^{C_1} e^{kt}$$

$$A = Ce^{kt}$$

$t=0$
$A=A_0$: $\qquad A_0 = C$

$$A = A_0 e^{kt}$$

→

$$A = A_0 e^{kt}$$

$t = 5600$
$A = 0.5A_0$: $0.5A_0 = A_0 e^{5600k}$

$$0.5 = e^{5600k}$$

$$\ln 0.5 = 5600k$$

$$k = \frac{\ln 0.5}{5600}$$

$$A = A_0 e^{\left(\frac{\ln 0.5}{5600}\right)t}$$

$A = 0.145A_0$: $0.145A_0 = A_0 e^{\left(\frac{\ln 0.5}{5600}\right)t}$

$$0.145 = e^{\frac{\ln 0.5}{5600}t}$$

$$\ln 0.145 = \frac{\ln 0.5}{5600} t$$

$$t = \frac{5600 \ln 0.145}{\ln 0.5}$$

$$\approx 15600 \text{ years}$$

(13) Let T = temperature (°C)
t = time (minutes)

$T(0) = 21°C \quad T_m = -12°C \quad T(0.5) = 10°C$

Find $T(1)$
and t so that $T(t) = -9°C$.

$$\frac{dT}{dt} = k(T - T_m)$$

$$\frac{dT}{dt} = k(T+12)$$

$$\frac{dT}{T+12} = k\,dt$$

$$\int \frac{dT}{T+12} = \int k\,dt$$

$$\ln|T+12| = kt + C_1$$

$$|T+12| = e^{kt+C_1}$$

$$|T+12| = e^{C_1} \cdot e^{kt}$$

$$T+12 = \pm e^{C_1} e^{kt}$$

$$T+12 = Ce^{kt}$$

$$T = -12 + Ce^{kt}$$

$t=0$, $T=21$:

$$21 = -12 + C$$

$$C = 33$$

$$T = -12 + 33e^{kt}$$

→

(13) Cont'd $\quad T = -12 + 33e^{kt}$

$T = 10$, $t = 0.5$: $\quad 10 = -12 + 33e^{k(0.5)}$

$$22 = 33e^{0.5k}$$

$$\frac{22}{33} = e^{0.5k}$$

$$\ln\left(\frac{2}{3}\right) = 0.5k$$

$$k = 2\ln\left(\frac{2}{3}\right)$$

$$T = -12 + 33e^{2\ln\left(\frac{2}{3}\right)t}$$

$T(1) \approx 2.67°C$

$T = -9$: $\quad -9 = -12 + 33e^{2\ln\left(\frac{2}{3}\right)t}$

$$3 = 33e^{2\ln\left(\frac{2}{3}\right)t}$$

$$\frac{1}{11} = e^{2\ln\left(\frac{2}{3}\right)t}$$

$$\ln\frac{1}{11} = 2\ln\left(\frac{2}{3}\right)t$$

$$t = \frac{\ln\left(\frac{1}{11}\right)}{2\ln\left(\frac{2}{3}\right)}$$

≈ 2.96 minutes

(16) We'll use Newton's Law of Cooling twice but let's derive the solution first.

$$\frac{dT}{dt} = k(T - T_m)$$

$$\frac{dT}{T - T_m} = k\,dt$$

$$\int \frac{dT}{T - T_m} = \int k\,dt$$

$$\ln|T - T_m| = kt + C_1$$

$$|T - T_m| = e^{kt + C_1}$$

$$|T - T_m| = e^{C_1} \cdot e^{kt}$$

$$T - T_m = \pm e^{C_1} e^{kt}$$

$$T - T_m = Ce^{kt}$$

$$\boxed{T = T_m + Ce^{kt}}$$

Phase 1: T = temperature (°C)
t = time (mins)

$T(0) = 100°C$ $T(1) = 90°C$ $T_m = 0°C$

Find $T(2)$

$T = 100$, $t = 0$: $100 = 0 + C$
$C = 100$

$$\boxed{T = 100e^{kt}}$$

→

(16) Cont'd

$T=90$, $t=1$: $\quad 90 = 100e^{k}$

$0.9 = e^{k}$

$\ln 0.9 = k$

$$\boxed{T = 100e^{(\ln 0.9)t}}$$

$t=2$: $\quad T = 100e^{2\ln 0.9}$

$= 81°C$ $\qquad$ Bar is moved to Container B

Phase 2: $\quad T$ = temperature (°C)

t = time in Container B (mins)

$T_m = 100°C$

$T(0) = 81°C$ from above

$T(1) = 91°C$ (temp. rises by 10°C in 1st minute)

Find t when $T = 99.9°C$ and add 2 mins to account for the time in Container A.

$$\boxed{T = T_m + Ce^{kt}}$$

$T=81$, $t=0$: $\quad 81 = 100 + C$

$C = -19$

$$\boxed{T = 100 - 19e^{kt}}$$

$T=91$, $t=1$: $\quad 91 = 100 - 19e^{k}$

$-9 = -19e^{k}$

$\frac{9}{19} = e^{k}$

→

$\ln\left(\frac{9}{19}\right) = k$

$$\boxed{T = 100 - 19\, e^{\ln\left(\frac{9}{19}\right)t}}$$

$T = 99.9$: $\quad 99.9 = 100 - 19\, e^{\ln\left(\frac{9}{19}\right)t}$

$$-0.1 = -19\, e^{\ln\left(\frac{9}{19}\right)t}$$

$$\frac{0.1}{19} = e^{\ln\left(\frac{9}{19}\right)t}$$

$$\ln\left(\frac{0.1}{19}\right) = \ln\left(\frac{9}{19}\right)t$$

$$t = \frac{\ln\left(\frac{0.1}{19}\right)}{\ln\left(\frac{9}{19}\right)}$$

$$\approx 7.02 \text{ mins}$$

This is the time in Container B.

Total time $\approx 2 + 7.02$ mins

≈ 9.02 mins

(19) $T_m = 21°C$ Let T = temperature (°C).

Let t_1 = time of discovery. Let t = time since death (in hours)

$T(t_1) = 29.4°C$

$T(t_1+1) = 26.7°C$

$T(0) = 37°C$

Find t_1.

$$\frac{dT}{dt} = k(T - T_m)$$

$$\frac{dT}{dt} = k(T-21)$$

$$\frac{dT}{T-21} = k\,dt$$

$$\int \frac{dT}{T-21} = \int k\,dt$$

$$\ln|T-21| = kt + C_1$$

$$|T-21| = e^{kt+C_1}$$

$$|T-21| = e^{C_1} \cdot e^{kt}$$

$$T-21 = \pm e^{C_1} e^{kt}$$

$$T-21 = Ce^{kt}$$

$$\boxed{T = 21 + Ce^{kt}}$$

$T=37$
$t=0$:

$$37 = 21 + C$$

$$C = 16$$

$$\boxed{T = 21 + 16e^{kt}} \quad \rightarrow$$

$$T = 21 \cancel{+} 16 e^{kt}$$

$T = 26.7$
$t = t_1 + 1$:

$$26.7 = 21 + 16 e^{k(t_1+1)}$$
$$5.7 = 16 e^{k(t_1+1)} \qquad (1)$$

$T = 29.4$
$t = t_1$:

$$29.4 = 21 + 16 e^{kt_1}$$
$$8.4 = 16 e^{kt_1} \qquad (2)$$

$(1) \div (2)$:

$$\frac{5.7}{8.4} = \frac{16 e^{k(t_1+1)}}{16 e^{kt_1}}$$

$$\frac{5.7}{8.4} = e^{k}$$

(2) :

$$8.4 = 16 e^{kt_1}$$
$$8.4 = 16 (e^{k})^{t_1}$$

Sub $\frac{5.7}{8.4} = e^k$:

$$8.4 = 16 \left(\frac{5.7}{8.4}\right)^{t_1}$$

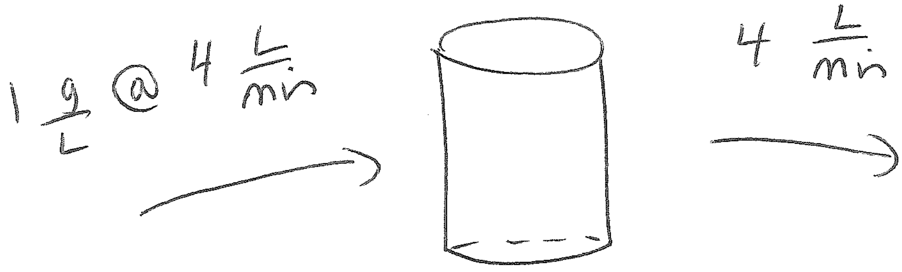
$$\frac{8.4}{16} = \left(\frac{5.7}{8.4}\right)^{t_1}$$

$$\ln\left(\frac{8.4}{16}\right) = \ln\left(\frac{5.7}{8.4}\right)^{t_1}$$

$$\ln\left(\frac{8.4}{16}\right) = t_1 \ln\left(\frac{5.7}{8.4}\right)$$

$$t_1 \simeq 1.7 \text{ hours}$$

(21)



Initially: 30 g salt
200 L fluid

Let $A =$ g of salt in tank
$t =$ time (min.).

$\boxed{\text{Volume is constant}}$

Inflow rate $= 1 \frac{g}{L} \cdot 4 \frac{L}{min} = 4 \frac{g}{min}$

Outflow rate $= \frac{A}{200} \frac{g}{L} \cdot 4 \frac{L}{min} = \frac{A}{50} \frac{g}{min}$

$$\frac{dA}{dt} = 4 - \frac{A}{50}$$

$$\frac{dA}{dt} + \frac{A}{50} = 4 \quad \text{Linear}$$

$$P(t) = \frac{1}{50} \quad I.F. = e^{\int \frac{1}{50} dt} = e^{\frac{t}{50}}$$

$$e^{\frac{t}{50}} \frac{dA}{dt} + e^{\frac{t}{50}} \frac{A}{50} = 4e^{\frac{t}{50}}$$

Integrate with respect to t:

$$e^{\frac{t}{50}} A = 4\left(50e^{\frac{t}{50}}\right) + C \quad \rightarrow$$

$$e^{\frac{t}{50}} A = 200 e^{\frac{t}{50}} + C$$

$$A = 200 + Ce^{-\frac{t}{50}}$$

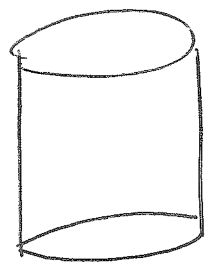
$A=30$, $t=0$: $\quad 30 = 200 + C$

$$C = -170$$

$$A = 200 - 170e^{-\frac{t}{50}}$$

(27)

$0.06 \frac{kg}{L}$ @ $22.68 \frac{L}{min}$ $\longrightarrow$



$\longrightarrow$ $15.12 \frac{L}{min}$

Initially: 4.55 kg salt
378 L fluid

Let A = kg of salt in tank
t = time (min.)

Volume is increasing by $22.68 \frac{L}{min} - 15.12 \frac{L}{min}$
$= 7.56 \frac{L}{min}$

$$\boxed{\text{Volume} = 378 + 7.56\,t}$$

$\rightarrow$

(27) Cont'd

$$\text{Inflow rate} = 0.06 \ \frac{kg}{L} \cdot 22.68 \ \frac{L}{min}$$

$$= 1.3608 \ \frac{kg}{min}$$

$$\text{Outflow rate} = \frac{A}{378+7.56t} \ \frac{kg}{L} \cdot 15.12 \ \frac{L}{min}$$

$$= \frac{15.12 A}{378+7.56t} \ \frac{kg}{min}$$

$$\frac{dA}{dt} = 1.3608 - \frac{15.12}{378+7.56t} A$$

$$\frac{dA}{dt} + \frac{15.12}{378+7.56t} A = 1.3608$$ Linear

$$P(t) = \frac{15.12}{378+7.56t} = 2\left(\frac{7.56}{378+7.56t}\right)$$

$$I.F. = e^{\int \frac{2(7.56)}{378+7.56t} dt}$$

$$= e^{2 \ln|378+7.56t|}$$

$$= e^{\ln|378+7.56t|^2}$$

$$= (378+7.56t)^2$$

$\rightarrow$

(27) Cont'd

$$(378+7.56t)^2\frac{dA}{dt}+15.12(378+7.56t)A=1.3608(378+7.56t)^2$$

Integrate with respect to t:

$$(378+7.56t)^2A=0.06(378+7.56t)^3+C$$

Let $u=378+7.56t$

$du=7.56\,dt$

$\int 1.3608(378+7.56t)^2dt$

$=\int \frac{1.3608\,u^2}{7.56}du$

$=0.06u^3+C_1$

$$A=0.06(378+7.56t)+C(378+7.56t)^{-2}$$

$A=4.55$, $t=0$:

$$4.55=0.06(378)+C(378)^{-2}$$

$$C=-2590486.92$$

$$A=0.06(378+7.56t)-2590486.92(378+7.56t)^{-2}$$

$t=30$: $A(30)\approx 29.21$ kg

(35) a) $m\frac{dv}{dt} = mg - kv \qquad v(0) = v_0$

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

$$\frac{dv}{dt} + \frac{k}{m}v = g \qquad \text{Linear}$$

$$P(t) = \frac{k}{m} \qquad \text{I.F.} = e^{\int \frac{k}{m} dt}$$

$$= e^{\frac{kt}{m}}$$

$$e^{\frac{kt}{m}}\frac{dv}{dt} + \frac{k}{m}e^{\frac{kt}{m}}v = ge^{\frac{kt}{m}}$$

Integrate with respect to t:

$$e^{\frac{kt}{m}}v = g\left(\frac{m}{k}e^{\frac{kt}{m}}\right) + C$$

$$v = \frac{mg}{k} + Ce^{-\frac{kt}{m}}$$

$v = v_0$, $t = 0$:

$$v_0 = \frac{mg}{k} + C$$

$$C = v_0 - \frac{mg}{k}$$

$$v = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-\frac{kt}{m}}$$

b) $\lim_{t\to\infty} v = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)(0)$

$$= \frac{mg}{k}$$

Meaning $v \to \frac{mg}{k}$ as $t \to \infty$.

(35) Cont'd

c) $$\frac{ds}{dt} = v$$

$$\frac{ds}{dt} = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-\frac{kt}{m}} \quad , \quad s(0) = 0$$

$$ds = \left[\frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-\frac{kt}{m}}\right]dt$$

$$\int ds = \int \left[\frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-\frac{kt}{m}}\right]dt$$

$$s = \frac{mg}{k}t - \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)e^{-\frac{kt}{m}} + C_1$$

$s=0$, $t=0$: $$0 = -\frac{m}{k}\left(v_0 - \frac{mg}{k}\right) + C_1$$

$$C_1 = \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)$$

$$s = \frac{mg}{k}t - \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)e^{-\frac{kt}{m}} + \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)$$