(1) $(x-y)\,dx + x\,dy = 0$

Homogeneous of degree 1

Subs $y = ux$

$dy = u\,dx + x\,du$

$$(x-ux)\,dx + x(u\,dx + x\,du) = 0$$

$$x\,dx - ux\,dx + ux\,dx + x^2\,du = 0$$

$$x\,dx = -x^2\,du$$

$$\frac{dx}{x} = -du$$

$$\int \frac{dx}{x} = \int -du$$

$$\ln|x| = -u + C_1$$

$$\ln|x| = -\left(\frac{y}{x}\right) + C_1$$

or $x\ln|x| = -y + Cx$

or $y + x\ln|x| = Cx$

(5) $(y^2 + yx)dx - x^2 dy = 0$

Homogeneous of degree 2

Sub $\begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases}$

$$(u^2x^2 + ux^2)dx - x^2(u\,dx + x\,du) = 0$$

$$u^2x^2dx + ux^2dx - ux^2dx - x^3du = 0$$

$$u^2x^2dx = x^3du$$

$$\frac{dx}{x} = u^{-2}du$$

$$\int \frac{dx}{x} = \int u^{-2}du$$

$$\ln|x| = -u^{-1} + C_1$$

$$\ln|x| = -\left(\frac{y}{x}\right)^{-1} + C_1$$

$$\ln|x| = -\frac{x}{y} + C_1$$

or $$y\ln|x| = -x + Cy$$

or $$x + y\ln|x| = Cy$$

7. $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$

$(y+x)\,dy = (y-x)\,dx$

Homogeneous of degree 1

Sub $\begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases}$

$(ux+x)(u\,dx + x\,du) = (ux - x)\,dx$

$u^2x\,dx + ux^2\,du + ux\,dx + x^2\,du = ux\,dx - x\,dx$

$u^2x\,dx + x\,dx + ux^2\,du + x^2\,du = 0$

$(u^2+1)\,x\,dx + (u+1)\,x^2\,du = 0$

$(u^2+1)\,x\,dx = -(u+1)\,x^2\,du$

$\dfrac{dx}{x} = \dfrac{-(u+1)}{u^2+1}\,du$

$\displaystyle\int \frac{dx}{x} = -\int\left[\frac{u}{u^2+1} + \frac{1}{u^2+1}\right]du$

Notice $\displaystyle\int \frac{2u}{u^2+1}\,du = \ln|u^2+1| + C_1$

so $\displaystyle\int \frac{u}{u^2+1}\,du = \frac{1}{2}\ln|u^2+1| + C_1$

$\rightarrow$

$$\ln|x| = -\left[\tfrac{1}{2}\ln|u^2+1| + \arctan u\right] + C_1$$

or $$\ln|x| = -\tfrac{1}{2}\ln\left|\tfrac{y^2}{x^2}+1\right| - \arctan\left(\tfrac{y}{x}\right) + C_1$$

or $$2\ln|x| = -\ln\left|\tfrac{y^2}{x^2}+1\right| - 2\arctan\left(\tfrac{y}{x}\right) + C_2$$

or $$\ln|x|^2 + \ln\left|\tfrac{y^2}{x^2}+1\right| + 2\arctan\left(\tfrac{y}{x}\right) = C_2$$

$$\boxed{\begin{array}{l} \text{Recall } \ln a + \ln b = \ln(ab) \\ \text{so } \ln|a| + \ln|b| = \ln|ab| \end{array}}$$

or $$\ln|y^2+x^2| + 2\arctan\left(\tfrac{y}{x}\right) = C$$

or $$\ln(y^2+x^2) + 2\arctan\left(\tfrac{y}{x}\right) = c$$

since $y^2+x^2 \geq 0$

(9) $-y\,dx + (x + \sqrt{xy})\,dy = 0$

Homogeneous of degree 1

Sub $\begin{cases} x = vy \\ dx = v\,dy + y\,dv \end{cases}$

$-y(v\,dy + y\,dv) + (vy + \sqrt{vy^2})\,dy = 0$

$-vy\,dy - y^2\,dv + vy\,dy + \sqrt{v}\,y\,dy = 0 \qquad (y>0)$

$-y^2\,dv = -\sqrt{v}\,y\,dy$

$y^2\,dv = \sqrt{v}\,y\,dy$

$v^{-1/2}\,dv = \frac{1}{y}\,dy$

$\int v^{-1/2}\,dv = \int \frac{1}{y}\,dy$

$2v^{1/2} = \ln|y| + C$

$2\sqrt{\frac{x}{y}} = \ln|y| + C$

or $4\frac{x}{y} = (\ln|y| + C)^2$

or $4x = y(\ln|y| + C)^2$

(11) $xy^2 \frac{dy}{dx} = y^3 - x^3$, $y(1) = 2$

Homogeneous of degree 3

$$xy^2\,dy = (y^3 - x^3)\,dx$$

Sub $\begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases}$

$$x u^2 x^2 (u\,dx + x\,du) = (u^3x^3 - x^3)\,dx$$

$$u^3x^3\,dx + u^2x^4\,du = u^3x^3\,dx - x^3\,dx$$

$$u^2x^4\,du = -x^3\,dx$$

$$u^2\,du = -\frac{1}{x}\,dx$$

$$\int u^2\,du = \int -\frac{1}{x}\,dx$$

$$\frac{u^3}{3} = -\ln|x| + C_1$$

$$u^3 = -3\ln|x| + C_2$$

$$\left(\frac{y}{x}\right)^3 = -3\ln|x| + C_2$$

$$y^3 = -3x^3\ln|x| + Cx^3$$

Sub $y=2$, $x=1$:

$$8 = C$$

$$y^3 = -3x^3\ln|x| + 8x^3$$

or $$y^3 + 3x^3\ln|x| = 8x^3$$

(13) $(x+ye^{y/x})dx - xe^{y/x}dy = 0 \quad , y(1)=0$

Ignore exponentials when calculating degrees.

Homogeneous of degree 1.

$$\text{Sub} \begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases} \qquad y/x = u$$

$$(x + uxe^{u})dx - xe^{u}(u\,dx + x\,du) = 0$$

$$x\,dx + uxe^{u}dx - uxe^{u}dx - x^2e^{u}du = 0$$

$$x\,dx = x^2e^{u}du$$

$$\frac{1}{x}dx = e^{u}du$$

$$\int \frac{1}{x}dx = \int e^{u}du$$

$$\ln|x| = e^{u} + C$$

$$\ln|x| = e^{y/x} + C$$

$\begin{matrix} y=0 \\ x=1 \end{matrix}$:

$$0 = 1 + C$$

$$C = -1$$

$$\ln|x| = e^{y/x} - 1$$

(17)

$$\frac{dy}{dx} = y(xy^3 - 1)$$

$$\frac{dy}{dx} = xy^4 - y$$

$$\frac{dy}{dx} + y = xy^4 \qquad \text{Bernoulli } n=4$$

$$\text{Sub} \begin{cases} y = u^{\frac{1}{1-n}} = u^{-\frac{1}{3}} \\ \dfrac{dy}{dx} = -\frac{1}{3} u^{-\frac{4}{3}} \dfrac{du}{dx} \end{cases}$$

$$-\frac{1}{3} u^{-\frac{4}{3}} \frac{du}{dx} + u^{-\frac{1}{3}} = x u^{-\frac{4}{3}}$$

Multiply by $-3u^{\frac{4}{3}}$:

$$\frac{du}{dx} - 3u = -3x \qquad \text{Linear}$$

$P(x) = -3$

$$\text{I.F.} = e^{\int -3\,dx}$$
$$= e^{-3x}$$

$$e^{-3x}\frac{du}{dx} - 3e^{-3x} u = -3xe^{-3x}$$

D	I
$-3x$ (+)	e^{-3x}
-3 (−)	$-\frac{1}{3}e^{-3x}$
	$\frac{1}{9}e^{-3x}$

Integrate with respect to x :

$$e^{-3x} u = xe^{-3x} + \frac{1}{3}e^{-3x} + C$$

$$u = x + \frac{1}{3} + Ce^{3x}$$

$$\boxed{\begin{matrix} y = u^{-1/3} \\ y^{-3} = u \end{matrix}}$$

$$y^{-3} = x + \frac{1}{3} + Ce^{3x}$$

(19)

$$t^2 \frac{dy}{dt} + y^2 = ty$$

$$\frac{dy}{dt} + t^{-2}y^2 = t^{-1}y$$

$$\frac{dy}{dt} - t^{-1}y = -t^{-2}y^2 \quad \text{Bernoulli } n=2$$

Sub $\begin{cases} y = u^{\frac{1}{1-n}} = u^{-1} \\ \frac{dy}{dt} = -u^{-2}\frac{du}{dt} \end{cases}$

$$-u^{-2}\frac{du}{dt} - t^{-1}u^{-1} = -t^{-2}u^{-2}$$

Multiply by $-u^2$:

$$\frac{du}{dt} + t^{-1}u = t^{-2} \quad \text{Linear}$$

$$p(t) = \frac{1}{t} \quad \text{I.F.} = e^{\int \frac{1}{t}dt}$$
$$= e^{\ln|t|}$$
$$= e^{\ln t} \quad (t>0)$$
$$= t$$

$$t\frac{du}{dt} + u = t^{-1}$$

Integrate with respect to t:

$$tu = \ln|t| + C_1$$
$$tu = \ln t + C_1 \quad (t>0)$$

$\boxed{\begin{matrix} y = u^{-1} \\ y^{-1} = u \end{matrix}}$

$$\frac{t}{y} = \ln t + C_1$$

or $e^{t/y} = e^{\ln t + C_1}$

or $e^{t/y} = e^{\ln t} \cdot e^{C_1}$ or $e^{t/y} = ct$

(23) $\frac{dy}{dx} = (x+y+1)^2$

Sub $u = x+y+1$

$$\begin{cases} \frac{du}{dx} = 1 + \frac{dy}{dx} \\ \frac{dy}{dx} = \frac{du}{dx} - 1 \end{cases}$$

$$\frac{du}{dx} - 1 = u^2$$

$$\frac{du}{dx} = u^2+1$$

$$\frac{du}{u^2+1} = dx$$

$$\int \frac{du}{u^2+1} = \int dx$$

$$\arctan u = x + C$$

$$u = \tan(x+c)$$

$$x+y+1 = \tan(x+c)$$

$$y = -x-1+\tan(x+c)$$

(25) $$\frac{dy}{dx} = \tan^2(x+y)$$

Sub $\begin{cases} u = x+y \\ \dfrac{du}{dx} = 1 + \dfrac{dy}{dx} \\ \dfrac{dy}{dx} = \dfrac{du}{dx} - 1 \end{cases}$

$$\frac{du}{dx} - 1 = \tan^2 u$$

$$\frac{du}{dx} = 1 + \tan^2 u$$

$$\frac{du}{dx} = \sec^2 u$$

$$\frac{du}{\sec^2 u} = dx$$

$$\cos^2 u \, du = dx$$

$$\frac{1+\cos 2u}{2} \, du = dx$$

$$\int \left(\frac{1}{2} + \frac{\cos 2u}{2}\right) du = \int dx$$

$$\frac{u}{2} + \frac{\sin 2u}{4} = x + C_1$$

$$2u + \sin 2u = 4x + C_2$$

$$2(x+y) + \sin(2x+2y) = 4x + C_2 \quad \text{or} \quad -2x+2y+\sin(2x+2y) = C$$

(27) $\frac{dy}{dx} = 2 + \sqrt{y-2x+3}$

Sub $\begin{cases} u = y - 2x + 3 \\ \frac{du}{dx} = \frac{dy}{dx} - 2 \\ \frac{dy}{dx} = \frac{du}{dx} + 2 \end{cases}$

$$\frac{du}{dx} + 2 = 2 + \sqrt{u}$$

$$\frac{du}{dx} = \sqrt{u}$$

$$u^{-1/2}\,du = dx$$

$$\int u^{-1/2}\,du = \int dx$$

$$2u^{1/2} = x + C$$

$$2\sqrt{y-2x+3} = x + C$$

or $4(y-2x+3) = (x+c)^2$