(3) $M_y = 4 \qquad N_x = 4$

$M_y = N_x \Rightarrow$ DE is exact

$f = \int (5x + 4y)\,dx$ AND $f = \int (4x - 8y^3)\,dy$

$= \frac{5x^2}{2} + 4xy + g(y)$ $\qquad = 4xy - 2y^4 + h(x)$

$\Rightarrow f = \frac{5x^2}{2} + 4xy - 2y^4$

DE : $df = 0$

Solution : $f = c$

$\frac{5x^2}{2} + 4xy - 2y^4 = c$

(5) $M_y = 4xy = N_x$

$\Rightarrow$ DE is exact

$f = \int (2xy^2 - 3)\,dx$ AND $f = \int (2x^2y + 4)\,dy$

$= x^2y^2 - 3x + g(y)$ $\qquad = x^2y^2 + 4y + h(x)$

$\Rightarrow f = x^2y^2 - 3x + 4y$

DE : $df = 0$

Solution : $f = c$

$x^2y^2 - 3x + 4y = c$

(7) $M_y = -2y \qquad N_x = 2x - 2y$

$M_y \neq N_x$

$\Rightarrow$ DE is not exact

(9) Rewrite DE : $(x - y^3 + y^2 \sin x)dx - (3xy^2 + 2y\cos x)\,dy = 0$

$M_y = -3y^2 + 2y\sin x$

$N_x = -3y^2 + 2y\sin x$

$M_y = N_x$

$\Rightarrow$ DE is exact

$f = \int (x - y^3 + y^2 \sin x)dx$ AND $f = \int -(3xy^2 + 2y\cos x)dy$

$= \frac{x^2}{2} - xy^3 - y^2\cos x + g(y) \qquad = \int [-3xy^2 - 2y\cos x]dy$

$= -xy^3 - y^2\cos x + h(x)$

Note : $g(y) = 0$

$\Rightarrow f = \frac{x^2}{2} - xy^3 - y^2\cos x$

DE : $df = 0$

Solution : $f = c$

$\frac{x^2}{2} - xy^3 - y^2\cos x = C$

Equivalently : $xy^3 + y^2\cos x - \frac{x^2}{2} = C$

(11) $M_y = y\left(\frac{1}{y}\right) + \ln y + xe^{-xy}$

$N_x = \ln y$

$M_y \neq N_x$

$\Rightarrow$ DE is not exact

(15) Rewrite:

$$\left(x^2y^3 - \frac{1}{1+9x^2}\right)dx + x^3y^2\,dy = 0$$

$$M_y = 3x^2y^2$$
$$N_x = 3x^2y^2$$

$M_y = N_x$

$\Rightarrow$ DE is exact

$$f = \int \left(x^2y^3 - \frac{1}{1+9x^2}\right)dx \quad \text{AND} \quad f = \int x^3y^2\,dy$$
$$= \frac{x^3y^3}{3} - \frac{1}{3}\arctan(3x) + g(y) \qquad = \frac{x^3y^3}{3} + h(x)$$

Let $u = 3x$
$du = 3dx$
$\frac{du}{3} = dx$

$$\int \frac{1}{1+9x^2}dx = \int \frac{1}{1+(3x)^2}dx$$
$$= \frac{1}{3}\int \frac{1}{1+u^2}du$$
$$= \frac{1}{3}\arctan u + C_1$$

$$\Rightarrow f = \frac{x^3y^3}{3} - \frac{1}{3}\arctan(3x)$$

DE: $df = 0$

Solution: $f = C$

$$\frac{x^3y^3}{3} - \frac{1}{3}\arctan(3x) = C \quad \text{or} \quad x^3y^3 - \tan^{-1}(3x) = C$$

(21) $(x+y)^2 dx + (2xy + x^2 - 1)dy = 0$, $y(1) = 1$

$M_y = 2(x+y)(1)$

$N_x = 2y + 2x$

$M_y = N_x \Rightarrow$ DE is exact

$f = \int (x+y)^2 dx$ AND $f = \int (2xy + x^2 - 1)dy$

$= \int (x^2 + 2xy + y^2) dx$ $= xy^2 + x^2 y - y + h(x)$

$= \frac{x^3}{3} + x^2 y + xy^2 + g(y)$

$\Rightarrow f = \frac{x^3}{3} + x^2 y + xy^2 - y$

DE : $df = 0$

Solution : $f = c$

$\frac{x^3}{3} + x^2 y + xy^2 - y = c$

Sub $x=1$, $y=1$: $\frac{1}{3} + 1 + 1 - 1 = c$

$c = \frac{4}{3}$

$\frac{x^3}{3} + x^2 y + xy^2 - y = \frac{4}{3}$

(23) $(4y+2t-5)dt + (6y+4t-1)dy = 0 \quad , \quad y(-1) = 2$

$$M_y = 4$$

$$N_t = 4$$

$M_y = N_t \Rightarrow$ DE is exact

$$f = \int (4y+2t-5)\,dt \quad \text{AND} \quad f = \int (6y+4t-1)\,dy$$

$$= 4ty + \underbrace{t^2 - 5t} + g(y) \qquad\qquad = 3y^2 + 4ty - y + h(t)$$

$$= \underbrace{3y^2 - y} + 4ty + h(t)$$

$$\Rightarrow f = 4ty + t^2 - 5t + 3y^2 - y$$

DE : $df = 0$

Solution : $f = c$

$$4ty + t^2 - 5t + 3y^2 - y = c$$

Sub $t = -1$, $y = 2$: $-8 + 1 + 5 + 12 - 2 = c$

$$c = 8$$

$$4ty + t^2 - 5t + 3y^2 - y = 8$$

(25) $(y^2\cos x - 3x^2y - 2x)dx + (2y\sin x - x^3 + \ln y)dy = 0$

$y(0) = e$

$M_y = 2y\cos x - 3x^2$

$N_x = 2y\cos x - 3x^2$

$M_y = N_x \Rightarrow$ DE is exact

$f = \int (y^2\cos x - 3x^2y - 2x)dx$ AND $f = \int (2y\sin x - x^3 + \ln y)dy$

$= y^2\sin x - x^3y - x^2 + g(y)$ $\qquad$ $= y^2\sin x - x^3y + y\ln y - y + h(x)$

To integrate $\ln y$:

D	I
$\ln y$	1
$\frac{1}{y}$	y

$\oplus$ $\ominus$

$\int \ln y\, dy = y\ln y - \int 1\, dy$

$= y\ln y - y + C_1$

$\Rightarrow f = y^2\sin x - x^3y - x^2 + y\ln y - y$

Solution: $f = C$

$y^2\sin x - x^3y - x^2 + y\ln y - y = C$

Sub $x=0$, $y=e$: $e\ln e - e = C$

$C = 0$

$y^2\sin x - x^3y - x^2 + y\ln y - y = 0$

(27) $M_y = 3y^2 + 4kxy^3$

$N_x = 3y^2 + 40xy^3$

$M_y = N_x \Rightarrow 4k = 40$

$k = 10$

(31) $M_y = 4y \qquad N_x = 2y$

$M_y \neq N_x \Rightarrow$ DE is not exact

$$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{2y}{2xy} = \frac{1}{x}$$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x| = x \quad (\text{for } x > 0)$$

Exact DE: $(2xy^2 + 3x^2)dx + (2x^2y)dy = 0$

$f = \int (2xy^2 + 3x^2)dx$ AND $f = \int 2x^2y\,dy$

$= x^2y^2 + x^3 + g(y) \qquad = x^2y^2 + h(x)$

(31) Cont'd $\Rightarrow f = x^2y^2 + x^3$

DE: $df = 0$

Solution: $f = C$

$x^2y^2 + x^3 = C$

(33) $M_y = 6x \qquad N_x = 18x$

$M_y \neq N_x \Rightarrow$ DE is not exact

$$\frac{N_x - M_y}{M} = \frac{12x}{6xy} = \frac{2}{y}$$

$$I.F. = e^{\int \frac{2}{y}\,dy} = e^{2\ln|y|} = e^{\ln|y|^2} = |y|^2 = y^2$$

Exact DE: $6xy^3\,dx + (4y^3 + 9x^2y^2)\,dy = 0$

$f = \int 6xy^3\,dx$ AND $f = \int (4y^3 + 9x^2y^2)\,dy$

$= 3x^2y^3 + g(y) \qquad = y^4 + 3x^2y^3 + h(x)$

$\Rightarrow f = 3x^2y^3 + y^4$

Solution: $f = C$

$3x^2y^3 + y^4 = C$

(35) $M_y = -6 \qquad N_x = 0$

$$\frac{M_y - N_x}{N} = \frac{-6}{-2}$$
$$= 3$$

$$\text{I.F.} = e^{\int 3\,dx}$$
$$= e^{3x}$$

Exact DE : $(10e^{3x} - 6ye^{3x} + 1)dx - 2e^{3x}dy = 0$

$f = \int (10e^{3x} - 6ye^{3x} + 1)dx$ AND $f = \int -2e^{3x}dy$

$= \frac{10}{3}e^{3x} - 2ye^{3x} + x + g(y) \qquad = -2ye^{3x} + h(x)$

$\Rightarrow f = \frac{10}{3}e^{3x} + x - 2ye^{3x}$

DE : $df = 0$

Solution : $f = C$

$$\frac{10}{3}e^{3x} + x - 2ye^{3x} = C$$

(37) $M_y = 0 \quad N_x = 2xy$

Two options:

$$\frac{M_y - N_x}{N} = \frac{-2xy}{x^2y + 4y}$$

$$= \frac{-2xy}{(x^2+4)y}$$

$$= \frac{-2x}{x^2+4}$$ is a function of x

$$\boxed{\frac{N_x - M_y}{M} = \frac{2xy}{x} = 2y \text{ is a function of } y}$$

This seems easier but both work.

$$I.F. = e^{\int 2y\,dy}$$

$$= e^{y^2}$$

$$xe^{y^2}dx + (x^2ye^{y^2} + 4ye^{y^2})dy = 0$$ is an exact DE

Recall $\int e^u du = e^u + C_1$

So $\int 2ye^{y^2}dy = e^{y^2} + C_1$

$\int ye^{y^2}dy = \frac{e^{y^2}}{2} + C_1$ →

$f = \int x e^{y^2} \, dx$ AND $f = \int (x^2 y e^{y^2} + 4y e^{y^2}) \, dy$

$= \frac{x^2 e^{y^2}}{2} + g(y)$ $= \frac{x^2 e^{y^2}}{2} + 2e^{y^2} + h(x)$

$\Rightarrow f = \frac{x^2 e^{y^2}}{2} + 2e^{y^2}$

DE : $df = 0$

Solution : $f = C$

$\frac{x^2 e^{y^2}}{2} + 2e^{y^2} = C$

Sub $y = 0$, $x = 4$: $8 + 2 = C$

$C = 10$

$\frac{x^2 e^{y^2}}{2} + 2e^{y^2} = 10$

or $x^2 e^{y^2} + 4e^{y^2} = 20$

or $(x^2 + 4) e^{y^2} = 20$