5 $\dfrac{dy}{dx} + 3x^2y = x^2$

$P(x) = 3x^2$

$I.F. = e^{\int 3x^2\,dx}$

$= e^{x^3}$

$e^{x^3}\dfrac{dy}{dx} + 3x^2e^{x^3}y = x^2e^{x^3}$

$\int\left(e^{x^3}\dfrac{dy}{dx} + 3x^2e^{x^3}y\right)dx = \int x^2e^{x^3}dx$

$u = x^3$
$du = 3x^2dx$
$\dfrac{du}{3} = x^2dx$
$\frac{1}{3}\int e^u\,du$
$= \frac{1}{3}e^u + C$

$e^{x^3}y = \frac{1}{3}e^{x^3} + C$

$y = \frac{1}{3} + Ce^{-x^3}$

The interval is $(-\infty, \infty)$.
Ce^{-x^3} is a transient term.

(7) $x^2 y' + xy = 1$

$$\frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x^2}$$

$$p(x) = \frac{1}{x}$$

$$\begin{aligned} \text{I.F.} &= e^{\int \frac{1}{x} dx} \\ &= e^{\ln|x|} \\ &= |x| \\ &= x \quad \text{on } (0,\infty) \end{aligned}$$

$$x \frac{dy}{dx} + y = \frac{1}{x}$$

$$\int \left(x \frac{dy}{dx} + y \right) dx = \int \frac{1}{x} dx$$

$$xy = \ln|x| + C$$

$$y = x^{-1} \ln|x| + Cx^{-1}$$

or $y = x^{-1} \ln x + C x^{-1}$

The interval is $(0,\infty)$.

Recall from Math 250A that $\lim_{x \to \infty} \frac{\ln x}{x} = 0$ by L'Hôpital's Rule.

Both terms are transient.

(9) $x\dfrac{dy}{dx} - y = x^2 \sin x$

$$\frac{dy}{dx} - \frac{1}{x}y = x\sin x$$

$$P(x) = -\frac{1}{x}$$

$$\text{I.F.} = e^{\int -\frac{1}{x}\,dx}$$
$$= e^{-\ln|x|}$$
$$= e^{\ln|x|^{-1}}$$
$$= |x|^{-1}$$
$$= x^{-1} \quad \text{on } (0,\infty)$$

$$x^{-1}\frac{dy}{dx} - x^{-2}y = \sin x$$

$$\int \left(x^{-1}\frac{dy}{dx} - x^{-2}y\right)dx = \int \sin x\,dx$$

$$x^{-1}y = -\cos x + C$$

$$y = Cx - x\cos x$$

The interval is $(0,\infty)$.
None of the terms are transient.

(11) $x\frac{dy}{dx} + 4y = x^3 - x$

$$\frac{dy}{dx} + \frac{4}{x}y = x^2 - 1$$

$$P(x) = \frac{4}{x}$$

$$\text{I.F.} = e^{\int \frac{4}{x}dx}$$
$$= e^{4\ln|x|}$$
$$= e^{\ln|x|^4}$$
$$= |x|^4$$
$$= x^4$$

<u>Note:</u> ln 0 is undefined, so the interval must exclude 0.

Interval will be $(0, \infty)$

$$x^4\frac{dy}{dx} + 4x^3y = x^6 - x^4$$

$$\int\left(x^4\frac{dy}{dx} + 4x^3y\right)dx = \int(x^6 - x^4)dx$$
$$x^4y = \frac{x^7}{7} - \frac{x^5}{5} + C$$
$$y = \frac{x^3}{7} - \frac{x}{5} + Cx^{-4}$$

Interval is $(0, \infty)$.

Cx^{-4} is a transient term.

(13) $x^2 y' + x(x+2)y = e^x$

$$\frac{dy}{dx} + \frac{x^2+2x}{x^2} y = x^{-2} e^x$$

$$\frac{dy}{dx} + \left(1 + \frac{2}{x}\right) y = x^{-2} e^x$$

$$P(x) = 1 + \frac{2}{x}$$

$$\text{I.F.} = e^{\int (1 + \frac{2}{x}) dx}$$

$$= e^{x + 2\ln|x|}$$

$$= e^x \cdot e^{2\ln|x|}$$

$$= e^x \cdot e^{\ln|x|^2}$$

$$= e^x |x|^2$$

$$= x^2 e^x$$

Note: ln 0 is undefined, so the interval must exclude 0. Interval will be $(0, \infty)$

$$x^2 e^x \frac{dy}{dx} + \left(1 + \frac{2}{x}\right) x^2 e^x y = e^{2x}$$

Integrate with respect to x:

$$x^2 e^x y = \frac{1}{2} e^{2x} + C$$

$$y = \frac{1}{2} x^{-2} e^x + C x^{-2} e^{-x}$$

The interval is $(0, \infty)$.

$Cx^{-2}e^{-x}$ is a transient term.

(15) $y\,dx - 4(x+y^6)\,dy = 0$

Not linear in y (due to the y^6 term).

$$y\frac{dx}{dy} - 4(x+y^6) = 0$$

$$\frac{dx}{dy} - \frac{4}{y}x - 4y^5 = 0$$

$$\frac{dx}{dy} - \frac{4}{y}x = 4y^5$$ Linear in x ✓

$$P(y) = -\frac{4}{y}$$

$$\text{I.F.} = e^{\int -\frac{4}{y}dy}$$
$$= e^{-4\ln|y|}$$
$$= e^{\ln|y|^{-4}}$$
$$= |y|^{-4}$$
$$= y^{-4}$$

Note: ln 0 is undefined, so the interval must exclude 0. The interval is $0 < y < \infty$

$$y^{-4}\frac{dx}{dy} - 4y^{-5}x = 4y$$

Integrate with respect to y:

$$y^{-4}x = 2y^2 + C$$
$$x = 2y^6 + Cy^4$$

Interval is $0 < y < \infty$

There are no transient terms.

(17) $\cos x \frac{dy}{dx} + (\sin x)y = 1$ Section 2.3

$$\frac{dy}{dx} + (\tan x)y = \sec x$$

$$P(x) = \tan x$$

$$\text{I.F.} = e^{\int \tan x\,dx}$$

$$= e^{\ln|\sec x|}$$

$$= |\sec x|$$

$$= \sec x$$

We need an interval where $\sec x > 0$

$$\Rightarrow \cos x > 0$$

$$\Rightarrow -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\sec x \frac{dy}{dx} + (\sec x \tan x)y = \sec^2 x$$

Integrate with respect to x:

$$(\sec x)y = \tan x + C$$

$$y = \frac{\tan x}{\sec x} + \frac{C}{\sec x}$$

$$y = \sin x + C\cos x$$

Interval : $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

There are no transient terms.

(19) $$(x+1)\frac{dy}{dx} + (x+2)y = 2xe^{-x}$$

$$\frac{dy}{dx} + \left(\frac{x+2}{x+1}\right)y = \frac{2x}{x+1}e^{-x}$$

Note: $$\frac{x+2}{x+1} = \frac{(x+1)+1}{x+1} = 1 + \frac{1}{x+1}$$

$$\frac{dy}{dx} + \left(1 + \frac{1}{x+1}\right)y = \frac{2x}{x+1}e^{-x}$$

$$P(x) = 1 + \frac{1}{x+1}$$

$$\begin{aligned} I.F. &= e^{\int \left(1 + \frac{1}{x+1}\right)dx} \\ &= e^{x + \ln|x+1|} \\ &= e^{x} \cdot e^{\ln|x+1|} \\ &= e^{x}|x+1| \\ &= (x+1)e^{x} \end{aligned}$$

Interval: $(-1, \infty)$

$$(x+1)e^{x}\frac{dy}{dx} + (x+1)\left(1+\frac{1}{x+1}\right)e^{x}y = 2x$$

Integrate with respect to x:

$$(x+1)e^{x}y = x^2 + C$$

or $$y = \frac{x^2 + C}{(x+1)e^{x}}$$

Interval: $(-1, \infty)$

Both terms are transient (in other words, the whole solution is transient).

(21) $\frac{dr}{d\theta} + r\sec\theta = \cos\theta$ Linear in r

$P(\theta) = \sec\theta$

$$\text{I.F.} = e^{\int \sec\theta\, d\theta}$$

$$= e^{\ln|\sec\theta + \tan\theta|}$$

$$= |\sec\theta + \tan\theta|$$

$$= \sec\theta + \tan\theta$$

Interval: $\sec\theta + \tan\theta > 0$

$$\frac{1+\sin\theta}{\cos\theta} > 0$$

The interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ works because $\cos\theta > 0$ and $1+\sin\theta > 0$ on this interval.

$$(\sec\theta + \tan\theta)\frac{dr}{d\theta} + (\sec^2\theta + \sec\theta\tan\theta)r = \cos\theta(\sec\theta + \tan\theta)$$

$$(\sec\theta + \tan\theta)\frac{dr}{d\theta} + (\sec^2\theta + \sec\theta\tan\theta)r = 1 + \sin\theta$$

Integrate with respect to θ:

$$(\sec\theta + \tan\theta)r = \theta - \cos\theta + C$$

or $r = \dfrac{\theta - \cos\theta + C}{\sec\theta + \tan\theta}$

The interval is $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

None of the terms are transient.

(23) $x\frac{dy}{dx} + (3x+1)y = e^{-3x}$

$$\frac{dy}{dx} + \left(3 + \frac{1}{x}\right)y = x^{-1}e^{-3x}$$

$$P(x) = 3 + \frac{1}{x}$$

$$I.F. = e^{\int (3 + \frac{1}{x})dx}$$

$$= e^{3x + \ln|x|}$$

$$= e^{3x} \cdot e^{\ln|x|}$$

$$= |x|e^{3x}$$

$$= xe^{3x}$$ Interval: $(0, \infty)$

$$xe^{3x}\frac{dy}{dx} + \left(3 + \frac{1}{x}\right)xe^{3x}y = 1$$

$$\int \left(xe^{3x}\frac{dy}{dx} + \left(3 + \frac{1}{x}\right)xe^{3x}y\right)dx = \int 1\,dx$$

$$xe^{3x}y = x + C$$

$$y = e^{-3x} + Cx^{-1}e^{-3x}$$

The interval is $(0, \infty)$

All terms are transient (the whole solution is transient).

(27) $xy' + y = e^x \quad , \quad y(1) = 2$

$$\frac{dy}{dx} + \frac{1}{x} y = x^{-1} e^x$$

$$P(x) = \frac{1}{x}$$

$$I.F. = e^{\int \frac{1}{x} dx}$$

$$= e^{\ln|x|}$$

$$= |x|$$

$$= x \qquad \text{Interval is } (0, \infty)$$

$$x \frac{dy}{dx} + y = e^x$$

$$\int \left(x \frac{dy}{dx} + y \right) dx = \int e^x dx$$

$$xy = e^x + C$$

Sub $y=2$, $x=1$: $\quad 2 = e + C$

$$C = 2 - e$$

$$xy = e^x + (2-e)$$

or $\quad y = x^{-1} e^x + \dfrac{2-e}{x}$

Interval is $(0, \infty)$

(29) $L\dfrac{di}{dt} + Ri = E \qquad i(0) = i_0$

L, R, E, i_0 are constants

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$ Linear in i

$$P(t) = \frac{R}{L}$$

$$\text{I.F.} = e^{\int \frac{R}{L}dt}$$
$$= e^{\frac{R}{L}t}$$

$$e^{\frac{Rt}{L}}\frac{di}{dt} + \frac{R}{L}e^{\frac{Rt}{L}}i = \frac{E}{L}e^{\frac{Rt}{L}}$$

Integrate with respect to t:

$$e^{\frac{Rt}{L}}i = \frac{E}{L}\left(\frac{L}{R}e^{\frac{Rt}{L}}\right) + C$$

$$e^{\frac{Rt}{L}}i = \frac{E}{R}e^{\frac{Rt}{L}} + C$$

or $i = \dfrac{E}{R} + Ce^{-\frac{Rt}{L}}$

Sub $i = i_0$, $t = 0$: $\quad i_0 = \dfrac{E}{R} + C$

$$C = i_0 - \frac{E}{R}$$

$$i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)e^{-\frac{Rt}{L}}$$

Interval is $\quad -\infty < t < \infty$

(33) $(x+1)\dfrac{dy}{dx} + y = \ln x \quad , \quad y(1) = 10$

$$\frac{dy}{dx} + \frac{1}{x+1}y = \frac{\ln x}{x+1}$$

$$P(x) = \frac{1}{x+1}$$

$$\text{I.F.} = e^{\int \frac{1}{x+1}\,dx}$$

$$= e^{\ln|x+1|}$$

$$= |x+1|$$

$$= (x+1)$$

Interval is $(-1, \infty)$ from the integrating factor

However due to the $\ln x$ term in the DE the interval is $(0, \infty)$.

$$(x+1)\frac{dy}{dx} + y = \ln x$$

Integration by Parts

D		I
$\ln x$	$+$	1
$\frac{1}{x}$	$-$	x

$$\ln x = x\ln x - \int 1\,dx$$
$$= x\ln x - x + C$$

Integrate with respect to x:

$$(x+1)y = x\ln x - x + C$$

$\rightarrow$

33 Cont'd

$$(x+1)y = x\ln x - x + C$$

Sub $y=20$, $x=1$: $20 = 0 - 1 + C$

$$C = 21$$

$$(x+1)y = x\ln x - x + 21$$

or $$y = \frac{x\ln x - x + 21}{x+1}$$

Interval is $(0,\infty)$