(5) $x \frac{dy}{dx} = 4y$

$$\frac{dy}{y} = \frac{4}{x}\,dx$$

$$\int \frac{dy}{y} = \int \frac{4}{x}\,dx$$

$$\ln|y| = 4\ln|x| + C_1$$

$$e^{\ln|y|} = e^{4\ln|x| + C_1}$$

$$|y| = e^{C_1} \cdot e^{4\ln|x|}$$

$$|y| = e^{C_1} \cdot e^{\ln|x|^4}$$

$$|y| = e^{C_1} \cdot |x|^4$$

$$y = \pm e^{C_1} x^4$$

$$y = Cx^4$$

(7) $\frac{dy}{dx} = e^{3x+2y}$

$\frac{dy}{dx} = e^{3x} \cdot e^{2y}$

$e^{-2y}\,dy = e^{3x}\,dx$

$\int e^{-2y}\,dy = \int e^{3x}\,dx$

$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + C_1$

$-3e^{-2y} = 2e^{3x} + C$

(9) $y \ln x \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$

$y \ln x\,dx = \frac{(y+1)^2}{x^2}\,dy$

$x^2 \ln x\,dx = \frac{(y+1)^2}{y}\,dy$

$x^2 \ln x\,dx = \frac{y^2+2y+1}{y}\,dy$

Section 2.2

$$x^2 \ln x \, dx = \left(y + 2 + \frac{1}{y}\right) dy$$

Integration by Parts

	D		I
	$\ln x$	$+$	x^2
	$\frac{1}{x}$	$-$	$\frac{x^3}{3}$

$$\int x^2 \ln x \, dx$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^2}{3} dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C_1$$

$$\frac{x^3}{3}\ln x - \frac{x^3}{9} = \frac{y^2}{2} + 2y + \ln|y| + C_2$$

or $6x^3 \ln x - 2x^3 = 9y^2 + 36y + 18\ln|y| + C$

We leave it in this form because it is too difficult to eliminate the $\ln|y|$ term.

(11) $\csc y\, dx + \sec^2 x\, dy = 0$

$\csc y\, dx = -\sec^2 x\, dy$

$\cos^2 x\, dx = -\sin y\, dy$

$\frac{1+\cos 2x}{2}\, dx = -\sin y\, dy$

$\int\left(\frac{1}{2} + \frac{\cos 2x}{2}\right)dx = -\int \sin y\, dy$

$\frac{x}{2} + \frac{\sin 2x}{4} = \cos y + C_1$

$2x + \sin 2x = 4\cos y + C_2$

or $4\cos y = 2x + \sin 2x + C$

(13) $(e^y+1)^2 e^{-y}\, dx + (e^x+1)^3 e^{-x}\, dy = 0$

$(e^y+1)^2 e^{-y}\, dx = -(e^x+1)^3 e^{-x}\, dy$

$(e^x+1)^{-3} e^x\, dx = -(e^y+1)^{-2} e^y\, dy$

$\int (e^x+1)^{-3} e^x\, dx = -\int (e^y+1)^{-2} e^y\, dy$

Sub $u = e^x + 1$ | Sub $v = e^y + 1$
$du = e^x\, dx$ | $dv = e^y\, dy$

→

$$\int u^{-3}\,du = -\int v^{-2}\,dv$$

$$-\frac{1}{2}u^{-2} = v^{-1} + C_1$$

$$-\frac{1}{2}(e^{x}+1)^{-2} = (e^{y}+1)^{-1} + C_1$$

$$-(e^{x}+1)^{-2} = 2(e^{y}+1)^{-1} + C_2$$

or $$-(e^{x}+1)^{-2} - 2(e^{y}+1)^{-1} = C_2$$

or $$(e^{x}+1)^{-2} + 2(e^{y}+1)^{-1} = C$$

(15)

$$\frac{dS}{dr} = kS$$

$$\frac{dS}{S} = k\,dr$$

$$\int \frac{dS}{S} = \int k\,dr$$

$$\ln|S| = kr + C_1$$

$$e^{\ln|S|} = e^{kr+C_1}$$

$$|S| = e^{C_1} \cdot e^{kr}$$

$$S = \pm e^{C_1} \cdot e^{kr}$$

$$S = Ce^{kr}$$

(17) $\frac{dP}{dt} = P - P^2$

$$\frac{dP}{P-P^2} = dt$$

$$\frac{dP}{P(1-P)} = dt$$

$$\int \frac{dP}{P(1-P)} = \int dt$$

Partial Fractions

$$\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P}$$

$$1 = A(1-P) + BP$$

Sub $P=0$: $1 = A$

Sub $P=1$: $1 = B$

$$\frac{1}{P(1-P)} = \frac{1}{P} + \frac{1}{1-P}$$

$$\int \left(\frac{1}{P} + \frac{1}{1-P}\right) dP = \int dt$$

Recall $\int \frac{dP}{a+bP} = \frac{1}{b} \ln|a+bP| + C_1$

$$\ln|P| - \ln|1-P| = t + C_2$$

$\rightarrow$

$$\ln\left|\frac{P}{1-P}\right| = t + C_2$$

$$e^{\ln\left|\frac{P}{1-P}\right|} = e^{t+C_2}$$

$$\left|\frac{P}{1-P}\right| = e^{C_2}\cdot e^{t}$$

$$\frac{P}{1-P} = \pm e^{C_2}\cdot e^{t}$$

$$\frac{P}{1-P} = Ce^{t}$$

or $\quad P = Ce^{t}(1-P)$

An explicit solution:

$$P = Ce^{t} - PCe^{t}$$

$$P + PCe^{t} = Ce^{t}$$

$$P(1+Ce^{t}) = Ce^{t}$$

$$P = \frac{Ce^{t}}{1+Ce^{t}}$$

(21) $\frac{dy}{dx} = x\sqrt{1-y^2}$

$\frac{dy}{\sqrt{1-y^2}} = x\,dx$

$\int \frac{dy}{\sqrt{1-y^2}} = \int x\,dx$

$\sin^{-1} y = \frac{x^2}{2} + C_1$

or $y = \sin\left(\frac{x^2}{2} + C_1\right)$

(23) $\frac{dx}{dt} = 4(x^2+1)$ $\qquad x\left(\frac{\pi}{4}\right) = 1$

$\frac{dx}{x^2+1} = 4dt$

$\int \frac{dx}{x^2+1} = \int 4dt$

$\tan^{-1} x = 4t + C_1$

$x=1$, $t=\frac{\pi}{4}$: $\tan^{-1} 1 = \pi + C_1$

$\frac{\pi}{4} = \pi + C_1$

$C_1 = \frac{-3\pi}{4}$

$$\tan^{-1} x = 4t - \frac{3\pi}{4}$$

or $$x = \tan\left(4t - \frac{3\pi}{4}\right)$$

(25) $$x^2 \frac{dy}{dx} = y - xy \qquad y(-1) = -1$$

$$x^2\, dy = y(1-x)\, dx$$

$$\frac{dy}{y} = \frac{1-x}{x^2}\, dx$$

$$\int \frac{dy}{y} = \int \left(\frac{1}{x^2} - \frac{1}{x}\right) dx$$

$$\ln|y| = -x^{-1} - \ln|x| + C_1$$

$$e^{\ln|y|} = e^{-x^{-1} - \ln|x| + C_1}$$

$$|y| = e^{C_1} \cdot e^{-x^{-1}} \cdot e^{-\ln|x|}$$

$$y = \pm e^{C_1} e^{-\frac{1}{x}} \cdot e^{\ln|x|^{-1}}$$

$$y = \pm e^{C_1} e^{-\frac{1}{x}} |x|^{-1}$$

$$y = \pm \frac{e^{C_1} e^{-\frac{1}{x}}}{x}$$

$$y = \frac{C e^{-\frac{1}{x}}}{x}$$

→

$y = -1$, $x = -1$: $-1 = \frac{Ce}{-1}$

$$C = \frac{1}{e}$$

$$y = \frac{\frac{1}{e} \cdot e^{-\frac{1}{x}}}{x}$$

$$y = \frac{e^{-1} \cdot e^{-\frac{1}{x}}}{x}$$

$$y = \frac{e^{-1 - \frac{1}{x}}}{x}$$

or $$y = \frac{e^{-(1 + \frac{1}{x})}}{x}$$