(1) $\dfrac{e^y\,dy}{dx} = \dfrac{2\sin 3x}{(e^y+1)^6}$

Separable

$(e^y+1)^6 e^y\,dy = 2\sin 3x\,dx$

$\int (e^y+1)^6 e^y\,dy = \int 2\sin 3x\,dx$

Let $I = \int (e^y+1)^6 e^y\,dy$

Sub $u = e^y+1$

$du = e^y dy$

$I = \int u^6\,du$

$= \frac{u^7}{7} + C$

$= \frac{1}{7}(e^y+1)^7 + C_1$

$\frac{1}{7}(e^y+1)^7 = -\frac{2}{3}\cos 3x + C_2$

$3(e^y+1)^7 = -14\cos 3x + C$

(2) $\frac{dy}{dx} = \frac{xy^4}{\sqrt{1+x^2}}$, $y(0) = \frac{1}{3}$

$y^{-4}dy = \frac{x\,dx}{\sqrt{1+x^2}}$ Separable

$\int y^{-4}dy = \int \frac{x}{\sqrt{1+x^2}}\,dx$

Let $I = \int \frac{x\,dx}{\sqrt{1+x^2}}$

Sub $u = 1+x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$I = \frac{1}{2}\int u^{-1/2}\,du$

$= u^{1/2} + C_1$

$= \sqrt{1+x^2} + C_1$

$-\frac{1}{3}y^{-3} = \sqrt{1+x^2} + C_1$

Sub $x=0$, $y=\frac{1}{3}$: $-9 = 1 + C_1$

$-10 = C_1$

$-\frac{1}{3}y^{-3} = \sqrt{1+x^2} - 10$

$y^{-3} = -3\sqrt{1+x^2} + 30$

$y = (30 - 3\sqrt{1+x^2})^{-1/3}$

(3) $\frac{dy}{dx} + \left(1 + \frac{2}{x}\right)y = 6$ **Linear**

$P(x) = 1 + \frac{2}{x}$

$e^{\int P(x)dx} = e^{\int \left(1+\frac{2}{x}\right)dx}$

$= e^{x + 2\ln|x|}$

$= e^{x} \cdot e^{2\ln|x|}$

$= e^{x} \cdot e^{\ln|x|^2}$

$= e^{x}\,|x|^2$

$= x^2 e^{x}$

$x^2 e^{x} \frac{dy}{dx} + (x^2 + 2x)e^{x} y = 6x^2 e^{x}$

Integrate with respect to x:

	D	I
⊕	$6x^2$	e^x
⊖	$12x$	e^x
⊕	12	e^x
	0	e^x

$x^2 e^{x} y = (6x^2 - 12x + 12)e^{x} + C$

$y = 6 - \frac{12}{x} + \frac{12}{x^2} + \frac{C}{x^2 e^{x}}$

(4) $y' - 4xy = \dfrac{e^{2x^2}}{x^2-4}$ **Linear**

$P(x) = -4x$

$e^{\int P(x)dx} = e^{\int -4x\,dx}$

$= e^{-2x^2}$

$e^{-2x^2}y' - 4xe^{-2x^2}y = \dfrac{1}{x^2-4}$

$e^{-2x^2}y' - 4xe^{-2x^2}y = \dfrac{1}{(x-2)(x+2)}$

Partial Fractions

$\dfrac{1}{(x-2)(x+2)} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$

$1 = A(x+2) + B(x-2)$

Sub $x=-2$: $1 = -4B \Rightarrow B = -\frac{1}{4}$

$x=2$: $1 = 4A \Rightarrow A = \frac{1}{4}$

$e^{-2x^2}y' - 4xe^{-2x^2}y = \frac{1}{4}\frac{1}{x-2} - \frac{1}{4}\frac{1}{x+2}$

Integrate with respect to x:

$e^{-2x^2}y = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$

$y = \frac{e^{2x^2}}{4}[\ln|x-2| - \ln|x+2|] + Ce^{2x^2}$