A1. [6 marks] Find the matrix P that diagonalizes $A = \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix}$.

$$\begin{vmatrix} \lambda-2 & 3 \\ -1 & \lambda-6 \end{vmatrix} = (\lambda-2)(\lambda-6)+3$$
$$= \lambda^2 - 8\lambda + 15$$

Set $\lambda^2 - 8\lambda + 15 = 0$

$(\lambda-3)(\lambda-5) = 0$

$\lambda = 3, 5$

E_3 : $[A - 3I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} -1 & -3 & 0 \\ 1 & 3 & 0 \end{array}\right]$$

$R_1 \div (-1)$, $R_2 + R_1$ $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -3t$, $x_2 = t$

$\vec{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} t$

Basis for E_3 $\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$

E_5 : $[A - 5I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$R_1 \div (-3)$, $R_2 + \frac{1}{3}R_1$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = -t$, $x_2 = t$

$\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$

Basis for E_5 $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$P = \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$$

A2. [4 marks] Use Question A1 to find A^5. Simplify your answer as much as possible.

$$P^{-1}AP = D$$

$$A = PDP^{-1}$$

$$A^5 = PD^5P^{-1}$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

$$D^5 = \begin{bmatrix} 3^5 & 0 \\ 0 & 5^5 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$$

$$A^5 = \frac{1}{2}\underbrace{\begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^5 & 0 \\ 0 & 5^5 \end{bmatrix}}\begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$$

[Easiest to put the $\frac{1}{2}$ in front.]

$$= \frac{1}{2}\begin{bmatrix} -3^6 & -5^5 \\ 3^5 & 5^5 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -2396 & -8646 \\ 2882 & 9132 \end{bmatrix}$$

$$= \begin{bmatrix} -1198 & -4323 \\ 1441 & 4566 \end{bmatrix}$$

Note: If you used $P = \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}$

then $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}$

A3. [5 marks] Find the algebraic and geometric multiplicities of each eigenvalue of $A = \begin{bmatrix} 4 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$.

Upper triangular

Characteristic polynomial is $(\lambda-4)^2(\lambda-1)$

Algebraic multiplicity of $\lambda=4$ is 2.

" $\lambda=1$ is 1.

E_4: $[A-4I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 0 & -3 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \div (-3)$, $R_2 - R_1$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \div 3$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} x_2=0 \\ x_3=0 \end{array}$$

$x_1=t$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t$$

E_1: $[A-I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 3 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$R_1 \div 3$, $R_2 \div 3$, $R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} x_1=t \\ x_3=0 \end{array}$$

$x_2=t$

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t$$

Geometric multiplicity of $\lambda=4$ is 1.

" $\lambda=1$ is 1.

A4. [3 marks] A is a 5×5 matrix with $\det A = 7$. Find $\det(-2A^{-1}A^TA^3)$.

$\det(-2A^{-1}A^TA^3)$

$= \det(-2A^{-1})\det(A^T)[\det(A)]^3$

using $\det(BC) = \det(B)\det(C)$ repeatedly

$= \det(-2A^{-1})[\det(A)]^4$

$\det(A^T) = \det(A)$

$= (-2)^5 \det(A^{-1})[\det(A)]^4$

$\det(kB) = k^n \det(B)$ when B is $n \times n$

$= (-2)^5 [\det(A)]^{-1} [\det(A)]^4$

$\det(A^{-1}) = \frac{1}{\det(A)}$ when $\det(A) \neq 0$

$= (-2)^5 (7)^3$

$= -10,976$

A5. [4 marks] **Find the values of x for which**

$$A = \begin{bmatrix} x & x & 0 \\ x & -6 & 1 \\ 3 & 5 & -2 \end{bmatrix} \text{ is invertible.}$$

$$|A| \neq 0$$

$$\begin{vmatrix} x & x & 0 \\ x & -6 & 1 \\ 3 & 5 & -2 \end{vmatrix} \neq 0$$

Expand along R_1

$$x \begin{vmatrix} -6 & 1 \\ 5 & -2 \end{vmatrix} - x \begin{vmatrix} x & 1 \\ 3 & -2 \end{vmatrix} \neq 0$$

$$x(7) \quad -x(-2x-3) \neq 0$$

$$7x + 2x^2 + 3x \neq 0$$

$$2x^2 + 10x \neq 0$$

$$2x(x+5) \neq 0$$

$$\boxed{x \neq 0, -5}$$

B1. [5 marks] Find det A using Gaussian Elimination.

$$A = \begin{bmatrix} 1 & 4 & 1 & 6 \\ 3 & -2 & 1 & 8 \\ 2 & 9 & 1 & 7 \\ 4 & 1 & -1 & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 4 & 1 & 6 \\ 3 & -2 & 1 & 8 \\ 2 & 9 & 1 & 7 \\ 4 & 1 & -1 & 0 \end{vmatrix}$$

$R_2 - 3R_1$, $R_3 - 2R_1$, $R_4 - 4R_1$

$$= \begin{vmatrix} 1 & 4 & 1 & 6 \\ 0 & -14 & -2 & -10 \\ 0 & 1 & -1 & -5 \\ 0 & -15 & -5 & -24 \end{vmatrix}$$

$R_2 \leftrightarrow R_3$

$$= -\begin{vmatrix} 1 & 4 & 1 & 6 \\ 0 & 1 & -1 & -5 \\ 0 & -14 & -2 & -10 \\ 0 & -15 & -5 & -24 \end{vmatrix}$$

$R_3 + 14R_2$, $R_4 + 15R_2$

$$= -\begin{vmatrix} 1 & 4 & 1 & 6 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & -16 & -80 \\ 0 & 0 & -20 & -99 \end{vmatrix}$$

$R_3 \div (-16)$

$$= +16 \begin{vmatrix} 1 & 4 & 1 & 6 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -20 & -99 \end{vmatrix}$$

$R_4 + 20R_3$

$$= 16 \begin{vmatrix} 1 & 4 & 1 & 6 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

upper triangular

$$= 16(1)(1)(1)(1)$$
$$= 16$$

B2. [3 marks] Solve for y in the system below using Cramer's Rule.

$$\begin{aligned} 5x+2y+z &= 0\\ x+6y+4z &= -11\\ -2x+3y-z &= -17\end{aligned}$$

$$y = \frac{|A_2|}{|A|}$$

$$= -4$$

$$|A_2| = \begin{vmatrix}5&0&1\\1&-11&4\\-2&-17&-1\end{vmatrix}$$

$$\begin{array}{ccccc} 5 & 0 & 1 & 5 & 0\\ 1 & -11 & 4 & 1 & -11\\ -2 & -17 & -1 & -2 & -17\end{array}$$

$$-(22 \quad -340 \quad 0) \quad (55 \quad 0 \quad -17)$$

$$|A_2| = 356$$

$$|A| = \begin{vmatrix}5&2&1\\1&6&4\\-2&3&-1\end{vmatrix}$$

$$\begin{array}{ccccc} 5 & 2 & 1 & 5 & 2\\ 1 & 6 & 4 & 1 & 6\\ -2 & 3 & -1 & -2 & 3\end{array}$$

$$-(-12 \quad 60 \quad -2) \quad (-30 \quad -16 \quad 3)$$

$$|A| = -89$$

B3. [3 marks] $\mathcal{B} = \left\{ \overset{\vec{v_1}}{\begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}}, \overset{\vec{v_2}}{\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}}, \overset{\vec{v_3}}{\begin{bmatrix} -25 \\ 13 \\ 2 \end{bmatrix}} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. Express $\mathbf{v} = \begin{bmatrix} 9 \\ 1 \\ -2 \end{bmatrix}$ as a linear combination of the three basis vectors.

Hint: There is an efficient method to solve the problem.

Orthogonal basis

$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

$$\vec{v} = \frac{\vec{v}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} + \frac{\vec{v}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2} + \frac{\vec{v}\cdot\vec{v_3}}{\vec{v_3}\cdot\vec{v_3}}\vec{v_3}$$

$$= \frac{-2}{38}\vec{v_1} + \frac{24}{21}\vec{v_2} - \frac{216}{798}\vec{v_3}$$

Simplified:

$$\vec{v} = \frac{-1}{19}\vec{v_1} + \frac{8}{7}\vec{v_2} - \frac{36}{133}\vec{v_3}$$

Check: $\frac{-1}{19}\begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} + \frac{8}{7}\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} - \frac{36}{133}\begin{bmatrix} -25 \\ 13 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ -2 \end{bmatrix}$ ✓

B4. [3 marks] Let $W = \text{span}\left(\begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -5 \\ 1 \end{bmatrix} \right)$. Find a basis for $W^{\perp}$.

$W = \text{row}(A) \qquad A = \begin{bmatrix} 1 & 4 & 0 & 1 \\ 2 & 8 & -5 & 1 \end{bmatrix}$

$W^{\perp} = \text{null}(A) \qquad \left[\begin{array}{cccc|c} 1 & 4 & 0 & 1 & 0 \\ 2 & 8 & -5 & 1 & 0 \end{array}\right]$

$R_2 - 2R_1 \qquad \left[\begin{array}{cccc|c} 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -5 & -1 & 0 \end{array}\right]$

$R_2 \div (-5) \qquad \left[\begin{array}{cccc|c} 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/5 & 0 \end{array}\right] \qquad x_1 = -4s - t, \quad x_3 = -1/5\, t$

$\uparrow x_2 = s \qquad \uparrow x_4 = t$

$$\vec{x} = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ -1/5 \\ 1 \end{bmatrix} t$$

Basis for $W^{\perp}$ $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1/5 \\ 1 \end{bmatrix} \right\}$

or $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \\ 5 \end{bmatrix} \right\}$

B5. [2 marks] Find a **nonzero** vector $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ so that the following set of vectors is orthogonal: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \right\}$.

Note: Many people did this by inspection. That's fine too.

First 3 vectors are orthogonal ✓

Need $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = 0 \longrightarrow \begin{cases} w = 0 \\ 2x + 3y + z = 0 \\ x - 2z = 0 \end{cases}$

Solve
$$\begin{array}{cccc|c} w & x & y & z & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_3$ $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 1 & 0 \end{array}\right]$

$R_3 - 2R_2$ $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 3 & 5 & 0 \end{array}\right]$

$R_3 \div 3$ $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 5/3 & 0 \end{array}\right]$ $x_1 = 0$, $x_2 = 2t$, $x_3 = -5/3\, t$

$x_4 = t$

$\vec{x} = \begin{bmatrix} 0 \\ 2 \\ -5/3 \\ 1 \end{bmatrix} t$

Choose any $\boxed{\text{nonzero } t}$

e.g. $t = 3$

$\vec{x} = \begin{bmatrix} 0 \\ 6 \\ -5 \\ 3 \end{bmatrix}$