Math 251 X02 Assignment Four

Name:______________________________

**Due: In class on Friday September 9**

Assignments must be completed on this paper. Marks may be deducted for not showing all your work.

1. [6 marks] a) Find an orthonormal basis for span $\left( \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right)$.

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\vec{v_2} = \begin{bmatrix} 1 \\ 5 \\ 6 \\ 0 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 1 \\ 5 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 5 \\ 6 \\ 0 \end{bmatrix} - \frac{6}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 2 \\ 6 \\ 0 \end{bmatrix} \qquad X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 6 \\ 0 \end{bmatrix} \right\}$$

$$\vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{12}{44} \begin{bmatrix} -2 \\ 2 \\ 6 \\ 0 \end{bmatrix} \qquad 11\vec{v_3} = \begin{bmatrix} 11 \\ 11 \\ 22 \\ 33 \end{bmatrix} - \begin{bmatrix} 11 \\ 11 \\ 0 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ 2 \\ 6 \\ 0 \end{bmatrix}$$

$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is an orthogonal basis. $= \begin{bmatrix} 6 \\ -6 \\ 4 \\ 33 \end{bmatrix}$

$$\text{orthonormal basis} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{11} \\ 1/\sqrt{11} \\ 3/\sqrt{11} \\ 0 \end{bmatrix}, \begin{bmatrix} 6/\sqrt{1177} \\ -6/\sqrt{1177} \\ 4/\sqrt{1177} \\ 33/\sqrt{1177} \end{bmatrix} \right\}$$

b) Use part a) to find the $QR$ factorization of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

$$Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{11} & 6/\sqrt{1177} \\ 1/\sqrt{2} & 1/\sqrt{11} & -6/\sqrt{1177} \\ 0 & 3/\sqrt{11} & 4/\sqrt{1177} \\ 0 & 0 & 33/\sqrt{1177} \end{bmatrix}$$

using $A = QR$ $\quad Q^T A = R$ since $Q^T Q = I$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{11} & 1/\sqrt{11} & 3/\sqrt{11} & 0 \\ 6/\sqrt{1177} & -6/\sqrt{1177} & 4/\sqrt{1177} & 33/\sqrt{1177} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{2} & 6/\sqrt{2} & 2/\sqrt{2} \\ 0 & 22/\sqrt{11} & 6/\sqrt{11} \\ 0 & 0 & 107/\sqrt{1177} \end{bmatrix}$$

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{11} & 6/\sqrt{1177} \\ 1/\sqrt{2} & 1/\sqrt{11} & -6/\sqrt{1177} \\ 0 & 3/\sqrt{11} & 4/\sqrt{1177} \\ 0 & 0 & 33/\sqrt{1177} \end{bmatrix} \begin{bmatrix} 2/\sqrt{2} & 6/\sqrt{2} & 2/\sqrt{2} \\ 0 & 22/\sqrt{11} & 6/\sqrt{11} \\ 0 & 0 & 107/\sqrt{1177} \end{bmatrix} \checkmark$$

2. [9 marks] Find the matrix $Q$ that orthogonally diagonalizes $A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$.

(Attach an extra page if necessary).

The characteristic polynomial of $A$ is $(\lambda-2)(\lambda-5)^2$.

$\lambda = 2, 5$ $\quad E_2$: $[A-2I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right]$$

$R_1 \Leftrightarrow R_2$ $$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right]$$

$R_2 + 2R_1$, $R_3 - R_1$ $$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$R_3 + R_2$ $$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-x_1 + 2x_2 - x_3 = 0$ $\quad x_1 = t$

$3x_2 - 3x_3 = 0$ $\quad x_2 = t$

REF

$x_3 = t$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$

Orthonormal Basis for $E_2 = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$

$E_5$ : $[A - 5I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right]$$

2. Cont'd

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad -x_1 - x_2 - x_3 = 0 \qquad x_1 = -s - t$$

REF

$x_2 = s \quad x_3 = t$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$$

Find orthonormal basis for $E_5$:

$$\vec{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \text{proj}_{\vec{v_1}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$2\vec{v_2} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Alternative basis for $E_5$:

$$\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right\}$$

orthonormal basis for $E_5$

$$= \left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

3. [4 marks] Express $z = \frac{x+yi}{13+5i}$ in the form $a + bi$.

$$z = \frac{(x+yi)}{(13+5i)} \cdot \frac{(13-5i)}{(13-5i)}$$

$$= \frac{(x+yi)(13-5i)}{169+25}$$

$$= \frac{13x+5y+(13y-5x)i}{194}$$

$$= \frac{13x+5y}{194} + \frac{13y-5x}{194}\, i$$

4. [6 marks] Find all the cube roots of $-8i$. Express each root in the form $a + bi$.

$$z = 8\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right]$$

Note: $|z|$ must be $\geq 0$.

$$z^{1/3} = 2\left[\cos\frac{\frac{3\pi}{2}+2\pi\alpha}{3} + i\sin\frac{\frac{3\pi}{2}+2\pi\alpha}{3}\right]$$

$\alpha = 0, 1, 2$

$$\frac{\frac{3\pi}{2}+2\pi\alpha}{3} = \frac{3\pi+4\pi\alpha}{6}$$

| $\alpha$ | $\frac{3\pi+4\pi\alpha}{6}$ |
|---|---|
| 0 | $\pi/2$ |
| 1 | $7\pi/6$ |
| 2 | $11\pi/6$ |

$$z_1^{1/3} = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 2i$$

$$z_2^{1/3} = 2\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right] = 2\left[-\frac{\sqrt{3}}{2} + i\left(\frac{-1}{2}\right)\right] = -\sqrt{3} - i$$

$$z_3^{1/3} = 2\left[\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right] = 2\left(\frac{\sqrt{3}}{2} + i\left(\frac{-1}{2}\right)\right] = \sqrt{3} - i$$