Math 251 X01 Assignment Three

Name:________________________________

Assignments must be completed on this paper. Marks may be deducted for not showing all your work.

1. [5 marks] Find all values of k for which the following matrix is invertible:

$$A = \begin{bmatrix} 2 & k & -8 \\ 1 & 4 & k \\ -2 & -2 & 6 \end{bmatrix}.$$

$$\det A = 2\begin{vmatrix} 4 & k \\ -2 & 6 \end{vmatrix} - k\begin{vmatrix} 1 & k \\ -2 & 6 \end{vmatrix} - 8\begin{vmatrix} 1 & 4 \\ -2 & -2 \end{vmatrix}$$

$$= 2(24+2k) - k(6+2k) - 8(-2+8)$$

$$= 48 + 4k - 6k - 2k^2 - 48$$

$$= -2k^2 - 2k$$

3

A is invertible when

$$\det A \neq 0$$

1

$$-2k^2 - 2k \neq 0$$

$$-2k(k+1) \neq 0$$

$$k \neq 0, -1$$

1

2. [8 marks] Find a matrix P that diagonalizes $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{bmatrix}$.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 3 & 2-\lambda & -3 \\ 1 & 0 & 1-\lambda \end{vmatrix}$$ (1)

$$= (2-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left((1-\lambda)^2 - 1\right)$$

$$= (2-\lambda)(\lambda^2 - 2\lambda)$$

$$= (2-\lambda)\lambda(\lambda-2)$$

$$= -(\lambda-2)^2\lambda$$ (1)

Set $|A - \lambda I| = 0$

$\lambda = 0, 2$ (1)

(1)

E_0 : $[A - 0I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 3 & 2 & -3 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$R_2 - 3R_1$, $R_3 - R_1$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2/2$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{x_3 = t}$

$x_1 + x_3 = 0$

$\boxed{x_1 = -t}$

$x_2 - 3x_3 = 0$

$\boxed{x_2 = 3t}$

Continued $\rightarrow$

$$\vec{x} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} t$$

Basis for $E_0 = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}$ 1

E_2 : $[A - 2I \mid \vec{0}\,]$ 1

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$$R_1/(-1) \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

$$\begin{array}{l} R_2 - 3R_1 \\ R_3 - R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{x_2 = s}$ $\boxed{x_3 = t}$

$x_1 - x_3 = 0$

$\boxed{x_1 = t}$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for $E_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ 1

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$ 1

Columns may appear in any order, and there are various choices for the basis vectors.

3. [4 marks] Find a basis for $W^\perp$ if

$$W = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } x = t, y = -4t, z = 2t \}.$$

$$W = \{ \begin{bmatrix} t \\ -4t \\ 2t \end{bmatrix} \}$$

Basis for $W = \{ \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \}$ 1

$W^\perp = \{ \vec{x}$ such that $\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \cdot \vec{x} = 0 \}$

$$\begin{array}{cccccc} & x_1 & x_2 & x_3 & & \\ \text{Solve} & [1 & -4 & 2 & | & 0] \end{array}$$ 1

$$\boxed{x_2 = s} \quad \boxed{x_3 = t}$$

$$x_1 - 4x_2 + 2x_3 = 0$$

$$\boxed{x_1 = 4s - 2t}$$

$$\vec{x} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for $W^\perp = \{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \}$ 2

4. [8 marks] Use the Gram-Schmidt process to find an orthogonal basis for

$$\text{span}\left(\begin{bmatrix}1\\6\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\3\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\\-1\end{bmatrix}\right).$$

$$\overline{v_1} = \begin{bmatrix}1\\6\\2\\1\end{bmatrix} \qquad X = \left\{\begin{bmatrix}1\\6\\2\\1\end{bmatrix}\right\} \text{ is a partial basis}$$

$$\overline{v_2} = \begin{bmatrix}1\\-2\\3\\-1\end{bmatrix} - \text{proj}_X \begin{bmatrix}1\\-2\\3\\-1\end{bmatrix}$$

$$\overline{v_2} = \begin{bmatrix}1\\-2\\3\\-1\end{bmatrix} - \frac{(-6)}{42}\begin{bmatrix}1\\6\\2\\1\end{bmatrix} \qquad 2$$

$$7\overline{v_2} = 7\begin{bmatrix}1\\-2\\3\\-1\end{bmatrix} + \begin{bmatrix}1\\6\\2\\1\end{bmatrix}$$

$$7\overline{v_2} = \begin{bmatrix}8\\-8\\23\\-6\end{bmatrix} \qquad X = \left\{\begin{bmatrix}1\\6\\2\\1\end{bmatrix}, \begin{bmatrix}8\\-8\\23\\-6\end{bmatrix}\right\}. \qquad 1$$

$$\overline{v_3} = \begin{bmatrix}2\\1\\-1\\-1\end{bmatrix} - \text{proj}_X \begin{bmatrix}2\\1\\-1\\-1\end{bmatrix}$$

$$\overline{v_3} = \begin{bmatrix}2\\1\\-1\\-1\end{bmatrix} - \frac{5}{42}\begin{bmatrix}1\\6\\2\\1\end{bmatrix} - \frac{(-9)}{693}\begin{bmatrix}8\\-8\\23\\-6\end{bmatrix} \qquad 3$$

$$\frac{-1}{77}$$

$$462\overline{v_3} = 462\begin{bmatrix}2\\1\\-1\\-1\end{bmatrix} - 55\begin{bmatrix}1\\6\\2\\1\end{bmatrix} + 6\begin{bmatrix}8\\-8\\23\\-6\end{bmatrix}$$

$$= \begin{bmatrix}917\\84\\-434\\-553\end{bmatrix} \quad 1 \qquad X = \left\{\begin{bmatrix}1\\6\\2\\1\end{bmatrix}, \begin{bmatrix}8\\-8\\23\\-6\end{bmatrix}, \begin{bmatrix}917\\84\\-434\\-553\end{bmatrix}\right\}$$

1