Math 251 X02 Assignment Three

Name:______________________________

Due: In class on Tuesday August 30
Assignments must be completed on this paper. Marks may be deducted for not showing all your work.

1. [4 marks] Find det A using Gaussian Elimination.

$$A = \begin{bmatrix} 2 & 1 & 6 & 9 \\ 2 & 1 & 3 & 0 \\ 6 & 8 & 1 & 0 \\ 4 & 12 & 1 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 6 & 9 \\ 2 & 1 & 3 & 0 \\ 6 & 8 & 1 & 0 \\ 4 & 12 & 1 & 1 \end{vmatrix}$$

$$= \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{array} \begin{vmatrix} 2 & 1 & 6 & 9 \\ 0 & 0 & -3 & -9 \\ 0 & 5 & -17 & -27 \\ 0 & 10 & -11 & -17 \end{vmatrix}$$

$$= \begin{array}{c} \\ \\ \\ R_2 \leftrightarrow R_3 \end{array} - \begin{vmatrix} 2 & 1 & 6 & 9 \\ 0 & 5 & -17 & -27 \\ 0 & 0 & -3 & -9 \\ 0 & 10 & -11 & -17 \end{vmatrix}$$

$$= \begin{array}{c} \\ \\ \\ R_4 - 2R_2 \end{array} - \begin{vmatrix} 2 & 1 & 6 & 9 \\ 0 & 5 & -17 & -27 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 23 & 37 \end{vmatrix}$$

$$= \begin{array}{c} \\ \\ \\ R_4 + \frac{23}{3}R_3 \end{array} - \begin{vmatrix} 2 & 1 & 6 & 9 \\ 0 & 5 & -17 & -27 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & -32 \end{vmatrix}$$

Upper triangular

$$= -(2)(5)(-3)(-32)$$

$$= -960$$

Try to write determinants rather than matrices.

2. [6 marks] Given $A = \begin{bmatrix} 2 & 1 & 1 & 6 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

a) Find the characteristic polynomial of A.

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -1 & -1 & -6 \\ 0 & \lambda-4 & -2 & 0 \\ 0 & 0 & \lambda-2 & -8 \\ 0 & 0 & 0 & \lambda-2 \end{vmatrix} = (\lambda-2)^3(\lambda-4)$$

b) Find a basis for the eigenspace E_2.

Solve $[A - 2I \mid \vec{0}]$

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 6 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 - 2R_1 \quad \left[\begin{array}{cccc|c} 0 & 1 & 1 & 6 & 0 \\ 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 + \frac{1}{2}R_2 \\ R_3 + \frac{2}{3}R_2 \end{array} \quad \left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = s \quad x_3 = t \\ x_2 = -t \\ x_4 = 0 \end{array}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} t$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

c) Explain why A is not diagonalizable.

Alg. mult. of $\lambda = 2$ is 3

but geo. mult. of $\lambda = 2$ is only 2.

3. [7 marks] Find the matrix P that diagonalizes $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix}$.

Show all your work.

A is upper triangular $\Rightarrow \lambda = 1, 0, -3$

E_1: $[A - I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$, $R_3 + 2R_1$

$$\left[\begin{array}{ccc|c} 0 & -1 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad x_2 = 0,\; x_3 = 0$$

$x_1 = t \qquad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t$

$E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

E_0: $[A - 0I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$R_3 + \frac{1}{2}R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad x_1 = 0,\; x_3 = 0$$

$x_2 = t \qquad \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t$

$E_0 = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

E_{-3}: $[A + 3I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$

$3x_2 + 6x_3 = 0$

$x_2 = -2t$

$4x_1 + 2x_3 = 0$

$x_1 = -\frac{1}{2}t$

$\vec{x} = \begin{bmatrix} -1/2 \\ -2 \\ 1 \end{bmatrix} t$

or $\begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix} t$

$E_{-3} = \operatorname{span}\left(\begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}\right)$

$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

or $P = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

4. [3 marks] Use the method involving the adjoint to find A^{-1}.

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 6 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} + 1\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix}$$
$$= 1(0) + 1(7)$$
$$= 7$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

Matrix of Cofactors $= \begin{bmatrix} 0 & 2 & -1 \\ -21 & 5 & 1 \\ 7 & -2 & 1 \end{bmatrix}$

transpose

$$\operatorname{adj}(A) = \begin{bmatrix} 0 & -21 & 7 \\ 2 & 5 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 0 & -21 & 7 \\ 2 & 5 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

Check: $A A^{-1} = I$ ✓

5. [2 marks] Let $W = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x = 5t, y = -2t, z = 10t\}$.

Find a basis for $W^\perp$.

W is a line through the origin. $W = \text{span}\left(\begin{bmatrix} 5 \\ -2 \\ 10 \end{bmatrix}\right)$

$\dim W = 1$

$\dim W + \dim W^\perp = 3$ so $\dim W^\perp = 2$

$W^\perp$ is a plane through the origin.

$\rightarrow$ Need 2 direction vectors orthogonal to $\begin{bmatrix} 5 \\ -2 \\ 10 \end{bmatrix}$

Many possibilities e.g $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$

Basis for $W^\perp = \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \right\}$

Alternatively: Could solve $\left[\begin{array}{ccc|c} 5 & \underset{\substack{\uparrow \\ x_2 = s}}{-2} & \underset{\substack{\uparrow \\ x_3 = t}}{10} & 0 \end{array}\right]$

$5x_1 = 2s - 10t$

$x_1 = 2/5\, s - 2t$

$\vec{x} = \begin{bmatrix} 2/5 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t$ Basis for $W^\perp = \left\{ \begin{bmatrix} 2/5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

6. [3 marks] $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} -69 \\ 45 \\ 42 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$.

Find $\mathbf{u}$ in $\mathbb{R}^3$ such that:

$$\mathbf{u} \cdot \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = -150, \quad \mathbf{u} \cdot \begin{bmatrix} 7 \\ 7 \\ 4 \end{bmatrix} = 342, \quad \mathbf{u} \cdot \begin{bmatrix} -69 \\ 45 \\ 42 \end{bmatrix} = 2850.$$

Given an orthogonal basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

$$\vec{u} = \frac{\vec{u}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} + \frac{\vec{u}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2} + \frac{\vec{u}\cdot\vec{v_3}}{\vec{v_3}\cdot\vec{v_3}}\vec{v_3}$$

$$= \frac{-150}{75}\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} + \frac{342}{114}\begin{bmatrix} 7 \\ 7 \\ 4 \end{bmatrix} + \frac{2850}{8550}\begin{bmatrix} -69 \\ 45 \\ 42 \end{bmatrix}$$

$\frac{1}{3}$

$$= -2\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 7 \\ 7 \\ 4 \end{bmatrix} + \begin{bmatrix} -23 \\ 15 \\ 14 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 46 \\ 12 \end{bmatrix}$$