Math 251 X02 Assignment Two

Name:________________________________

Due: In class on Tuesday August 9
Assignments must be completed on this paper. Marks may be deducted for not showing all your work.

1. [5 marks] a) Find the inverse of the matrix below using Gauss-Jordan Elimination.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 8 & -6 & -4 \\ -4 & 3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 8 & -6 & -4 & 0 & 1 & 0 \\ -4 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-8R_1 \\ R_3+4R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & -22 & -44 & -8 & 1 & 0 \\ 0 & 11 & 21 & 4 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2\div(-22) \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{8}{22} & \frac{-1}{22} & 0 \\ 0 & 11 & 21 & 4 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1-2R_2 \\ \\ R_3-11R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{6}{22} & \frac{2}{22} & 0 \\ 0 & 1 & 2 & 8/22 & \frac{-1}{22} & 0 \\ 0 & 0 & -1 & 0 & \frac{11}{22} & 1 \end{array}\right] \begin{array}{l} \Leftarrow 1-\frac{16}{22}=\frac{6}{22} \\ \\ \Leftarrow 4-\frac{88}{22}=0 \end{array}$$

$$\begin{array}{l} \\ \\ R_3\div(-1) \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 6/22 & 2/22 & 0 \\ 0 & 1 & 2 & 8/22 & -1/22 & 0 \\ 0 & 0 & 1 & 0 & \frac{-11}{22} & -1 \end{array}\right]$$

$$\begin{array}{l} R_1-R_3 \\ R_2-2R_3 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{6}{22} & \frac{13}{22} & 1 \\ 0 & 1 & 2 & 8/22 & \frac{21}{22} & 2 \\ 0 & 0 & 1 & 0 & \frac{-11}{22} & -1 \end{array}\right]$$

Clearest way to write A^{-1}:

$$A^{-1} = \frac{1}{22}\begin{bmatrix} 6 & 13 & 22 \\ 8 & 21 & 44 \\ 0 & -11 & -22 \end{bmatrix}$$

Check $A^{-1} \cdot A = I_3$ ✓

b) Use part a) to solve the system below.

$$\begin{aligned} x + 2y + 5z &= 5 \\ 8x - 6y - 4z &= -70 \\ -4x + 3y + z &= 33 \end{aligned}$$

$A\vec{x} = b$

$$\begin{bmatrix} 1 & 2 & 5 \\ 8 & -6 & -4 \\ -4 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -70 \\ 33 \end{bmatrix}$$

$\vec{x} = A^{-1} b$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 6 & 13 & 22 \\ 8 & 21 & 44 \\ 0 & -11 & -22 \end{bmatrix} \begin{bmatrix} 5 \\ -70 \\ 33 \end{bmatrix}$$

$$= \frac{1}{22} \begin{bmatrix} -154 \\ 22 \\ 44 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 1 \\ 2 \end{bmatrix} \checkmark$$

2. [3 marks] A square matrix A is called **skew symmetric** if $A^T = -A$. Show that if B is an $n \times n$ matrix then $C = B - B^T$ is skew symmetric.

$$\begin{aligned} C^T &= (B - B^T)^T \\ &= B^T - (B^T)^T \\ &= B^T - B \\ &= -(B - B^T) \\ &= -C \end{aligned}$$

3. [4 marks] Find the LU factorization of the matrix below.

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ -8 & 1 & 0 & 3 \\ 6 & 3 & 1 & 10 \\ 4 & -2 & 2 & 0 \end{bmatrix}$$

Do operations $R_i - kR_j$ only
Record k-value

$$\begin{array}{l} \\ R_2+4R_1 \\ R_3-3R_1 \\ R_4-2R_1 \end{array} \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 4 & 7 \\ 0 & 3 & -2 & 7 \\ 0 & -2 & 0 & -2 \end{bmatrix} \begin{array}{l} \\ k=-4 \\ k=3 \\ k=2 \end{array}$$

$$\begin{array}{l} \\ \\ R_3-3R_2 \\ R_4+2R_2 \end{array} \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -14 & -14 \\ 0 & 0 & 8 & 12 \end{bmatrix} \begin{array}{l} \\ \\ k=3 \\ k=-2 \end{array}$$

$$R_4+\tfrac{8}{14}R_3 \quad \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -14 & -14 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad k=-\tfrac{8}{14}=-\tfrac{4}{7}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 2 & -2 & -\tfrac{4}{7} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -14 & -14 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad A=LU$$

Check ✓

4. [3 marks] Use Question 3 to solve the system below.

$$\begin{aligned} 2x_1 \quad\quad + x_3 + \quad x_4 &= 1 \\ -8x_1 + x_2 \quad\quad + 3x_4 &= 19 \\ 6x_1 + 3x_2 + x_3 + 10x_4 &= 30 \\ 4x_1 - 2x_2 + 2x_3 \quad\quad &= -12 \end{aligned}$$

$A\vec{x} = b$

$LU\vec{x} = b$

Let $\vec{y} = U\vec{x}$ and solve $L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 2 & -2 & -\frac{4}{7} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 19 \\ 30 \\ -12 \end{bmatrix}$$

$y_1 = 1$

$-4y_1 + y_2 = 19$ $\quad y_2 = 23$

$3y_1 + 3y_2 + y_3 = 30$ $\quad y_3 = -42$

$2y_1 - 2y_2 - \frac{4}{7}y_3 + y_4 = -12$

$2 - 46 + 24 + y_4 = -12$

$y_4 = 8$

Now solve $U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -14 & -14 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 23 \\ -42 \\ 8 \end{bmatrix}$$

$4x_4 = 8$ $\quad x_4 = 2$

$-14x_3 - 14(2) = -42$ $\quad x_3 = 1$

$x_2 + 4(1) + 7(2) = 23$ $\quad x_2 = 5$

$2x_1 + 1 + 2 = 1$ $\quad x_1 = -1$

$$\vec{x} = \begin{bmatrix} -1 \\ 5 \\ 1 \\ 2 \end{bmatrix}$$

Check ✓

5. [3 marks] Show that the set S of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x = 4y$ and $z = -7y$ is a subspace of $\mathbb{R}^3$.

$$S = \left\{ \begin{bmatrix} 4y \\ y \\ -7y \end{bmatrix} \text{ where } y \text{ is a real } \# \right\}$$

1) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in S ✓

2) Let $\vec{u} = \begin{bmatrix} 4a \\ a \\ -7a \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4b \\ b \\ -7b \end{bmatrix}$ in S.

$$\vec{u} + \vec{v} = \begin{bmatrix} 4a+4b \\ a+b \\ -7a-7b \end{bmatrix} = \begin{bmatrix} 4(a+b) \\ a+b \\ -7(a+b) \end{bmatrix} \text{ in } S \checkmark$$

3) Let c be a scalar

$$c\vec{u} = \begin{bmatrix} 4ca \\ ca \\ -7ca \end{bmatrix} = \begin{bmatrix} 4(ca) \\ ca \\ -7(ca) \end{bmatrix} \text{ in } S \checkmark$$

6. [4 marks] Find a basis for the row space of A consisting of rows of A.

$$A = \begin{bmatrix} 1 & 2 & -1 & 9 \\ -2 & -3 & 1 & 6 \\ -1 & 1 & -2 & 63 \\ 4 & 3 & 3 & 3 \end{bmatrix}$$

Put rows → columns

$$A^T = \begin{bmatrix} 1 & -2 & -1 & 4 \\ 2 & -3 & 1 & 3 \\ -1 & 1 & -2 & 3 \\ 9 & 6 & 63 & 3 \end{bmatrix}$$

Locate leading entries in REF

$$\begin{matrix} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 9R_1 \end{matrix} \quad \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & 3 & -5 \\ 0 & -1 & -3 & 7 \\ 0 & 24 & 72 & -33 \end{bmatrix}$$

$$\begin{matrix} \\ \\ R_3 + R_2 \\ R_4 - 24R_2 \end{matrix} \quad \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 87 \end{bmatrix}$$

$$\begin{matrix} \\ \\ \\ R_4 - \frac{87}{2}R_3 \end{matrix} \quad \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Use columns 1, 2 and 4 of A^T.

Use rows 1, 2 and 4 of A.

Basis = $\{[1\ 2\ -1\ 9], [-2\ -3\ 1\ 6], [4\ 3\ 3\ 3]\}$

7. [3 marks] Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 12 \\ 13 \\ -6 \end{bmatrix}$.

Find $T(\mathbf{v})$ for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ with:

$$T(\mathbf{v}_1) = \begin{bmatrix} -6 \\ 30 \\ 24 \\ 8 \end{bmatrix}, T(\mathbf{v}_2) = \begin{bmatrix} -2 \\ 8 \\ 6 \\ 2 \end{bmatrix} \text{ and } T(\mathbf{v}_3) = \begin{bmatrix} -1 \\ 9 \\ 6 \\ 0 \end{bmatrix}.$$

Write $\vec{v}$ as a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

$$\begin{array}{ccc} a & b & c \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 13 \\ 7 & 2 & 2 & -6 \end{array}\right]$$

$$R_3 - 7R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 13 \\ 0 & 2 & 2 & -90 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 13 \\ 0 & 2 & 0 & -116 \end{array}\right] \quad \begin{array}{l} a = 12 \\ c = 13 \\ 2b = -116 \quad b = -58 \end{array}$$

$$\begin{aligned} T(\vec{v}) &= T(12\vec{v_1} - 58\vec{v_2} + 13\vec{v_3}) \\ &= T(12\vec{v_1}) + T(-58\vec{v_2}) + T(13\vec{v_3}) \\ &= 12T(\vec{v_1}) - 58T(\vec{v_2}) + 13T(\vec{v_3}) \end{aligned} \Big\} \text{ T is linear}$$

$$= 12\begin{bmatrix} -6 \\ 30 \\ 24 \\ 8 \end{bmatrix} - 58\begin{bmatrix} -2 \\ 8 \\ 6 \\ 2 \end{bmatrix} + 13\begin{bmatrix} -1 \\ 9 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 31 \\ 13 \\ 18 \\ -20 \end{bmatrix}$$