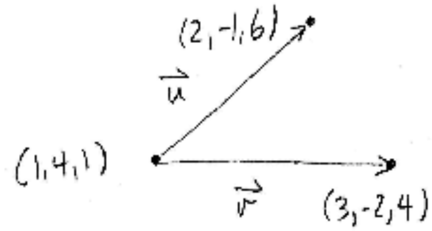
Math 251 X02 Assignment One

Name:________________________________

Due: In class on Tuesday July 19
Assignments must be completed on this paper. Marks may be deducted for not showing all your work.

1. [4 marks] **Find the area of the triangle with vertices** $(1, 4, 1)$, $(2, -1, 6)$ and $(3, -2, 4)$.



$\vec{u} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$

$\text{Area} = \frac{1}{2} \| \vec{u} \times \vec{v} \|$

$\begin{array}{ccccc} 1 & -5 & 5 & 1 & -5 \\ 2 & -6 & 3 & 2 & -6 \end{array}$

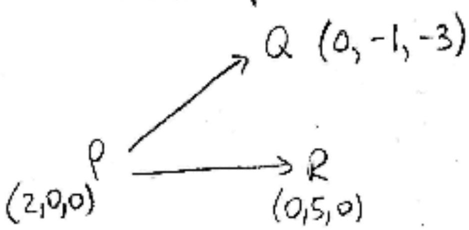
$\vec{u} \times \vec{v} = [15, 7, 4]$

$\| \vec{u} \times \vec{v} \| = \sqrt{15^2 + 7^2 + 4^2} = \sqrt{290}$

$\boxed{\text{Area} = \frac{\sqrt{290}}{2}}$

2. [3 marks] Find the vector form of the plane $5x + 2y - 4z = 10$.

Need 3 points on plane



$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$

$\vec{u} = \vec{PQ} = \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix}$

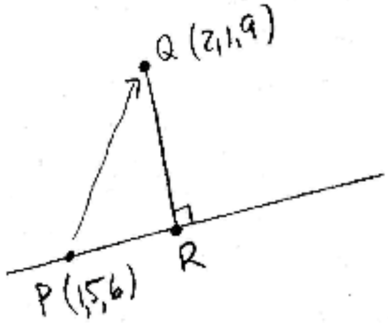
$\vec{v} = \vec{PR} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$

not scalar multiples (different directions) ✓

$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$

MANY ANSWERS ARE POSSIBLE

3. [4 marks] Line $\mathcal{L}$ is given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$. Find the point R on $\mathcal{L}$ that is closest to $Q = (2, 1, 9)$.



ℓ $\quad \overrightarrow{d}$ gives direction of ℓ $\quad \vec{d} = \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$

$PQ = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$

$$\overrightarrow{PR} = \operatorname{proj}_{\vec{d}} \overrightarrow{PQ} = \frac{\vec{d} \cdot \overrightarrow{PQ}}{\|\vec{d}\|^2} \vec{d} = \frac{9}{25} \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$$

To find R, we start at P and travel in direction $\overrightarrow{PR}$

$$R = (1,5,6) + \frac{9}{25}(-3,0,4) = \left(\frac{-2}{25}, 5, \frac{186}{25} \right)$$

4. [3 marks] Find the point of intersection of the lines given by $\mathbf{x} = \begin{bmatrix} 1 \\ -6 \\ 5 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ 13 \end{bmatrix}$, or show that there is no intersection.

Solve $\begin{bmatrix} 1 \\ -6 \\ 5 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ 13 \end{bmatrix}$

$$\begin{bmatrix} 1-s \\ -6+s \\ 5+4s \end{bmatrix} = \begin{bmatrix} 3-4t \\ -2+t \\ 3+13t \end{bmatrix}$$

$1-s=3-4t \qquad -s+4t=2 \quad ①$

$-6+s=-2+t \qquad s-t=4 \quad ②$

$5+4s=3+13t \qquad 4s-13t=-2 \quad ③$

$$\begin{array}{cc|c} s & t & \\ \end{array}$$
$$\left[\begin{array}{cc|c} -1 & 4 & 2 \\ 1 & -1 & 4 \\ 4 & -13 & -2 \end{array}\right]$$

$$\begin{array}{l} \\ R_1+R_2 \\ 4R_1+R_3 \end{array} \left[\begin{array}{cc|c} -1 & 4 & 2 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{array}\right] \qquad \begin{array}{l} -s+4t=2 \quad s=6 \\ t=2 \end{array}$$

Point of intersection is $\begin{bmatrix} 1 \\ -6 \\ 5 \end{bmatrix} + 6 \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 29 \end{bmatrix}$

$P = (-5, 0, 29)$

5. [4 marks] Solve using either Gaussian or Gauss-Jordan Elimination.

$$
\begin{aligned}
a+b+c+d &= 4\\
a+2b+3c+d &= 14\\
2a+2b+3c+3d &= 9\\
7a+8b+10c+8d &= 39
\end{aligned}
$$

$$
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 4\\
1 & 2 & 3 & 1 & 14\\
2 & 2 & 3 & 3 & 9\\
7 & 8 & 10 & 8 & 39
\end{array}\right]
$$

$$
\begin{array}{c}
\\
R_2-R_1\\
R_3-2R_1\\
R_4-7R_1
\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 4\\
0 & 1 & 2 & 0 & 10\\
0 & 0 & 1 & 1 & 1\\
0 & 1 & 3 & 1 & 11
\end{array}\right]
$$

$$
\begin{array}{c}
R_1-R_2\\
\\
\\
R_4-R_2
\end{array}
\left[\begin{array}{cccc|c}
1 & 0 & -1 & 1 & -6\\
0 & 1 & 2 & 0 & 10\\
0 & 0 & 1 & 1 & 1\\
0 & 0 & 1 & 1 & 1
\end{array}\right]
$$

$$
\begin{array}{c}
\\
R_1+R_3\\
R_2-2R_3\\
\\
R_4-R_3
\end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & c & d & \end{array}\\
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 2 & -5\\
0 & 1 & 0 & -2 & 8\\
0 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\quad \text{RREF}
$$

I find G-J is usually faster than doing back-sub.

$\boxed{d=t}$

$c+d=1$ $\quad\boxed{c=1-t}$

$b-2d=8$ $\quad\boxed{b=8+2t}$

$a+2d=-5$ $\quad\boxed{a=-5-2t}$

Final Answer:

$$
\begin{bmatrix} a\\ b\\ c\\ d \end{bmatrix} = \begin{bmatrix} -5\\ 8\\ 1\\ 0 \end{bmatrix} + t\begin{bmatrix} -2\\ 2\\ -1\\ 1 \end{bmatrix}
$$

6. [3 marks] Show that the span of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 15 \end{bmatrix}$ is all of $\mathcal{R}^2$.

Show that for any $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$
there is a solution x, y to:

$$x \begin{bmatrix} -1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -4 \\ 15 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Solve
$$\begin{array}{cc} x & y \end{array}$$
$$\left[\begin{array}{cc|c} -1 & -4 & a \\ 3 & 15 & b \end{array}\right]$$

$$R_2+3R_1 \quad \left[\begin{array}{cc|c} -1 & -4 & a \\ 0 & 3 & b+3a \end{array}\right] \quad \text{REF}$$

$3y = b+3a \qquad y = \dfrac{b+3a}{3}$

$-x-4y = a \qquad x = -a-4y$

$x = -a - \frac{4}{3}(b+3a)$

$x = -5a - \frac{4}{3}b$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5a - \frac{4}{3}b \\ \frac{b+3a}{3} \end{bmatrix} \checkmark$$

Check: $\left(-5a - \frac{4}{3}b\right)\begin{bmatrix} -1 \\ 3 \end{bmatrix} + \frac{b+3a}{3}\begin{bmatrix} -4 \\ 15 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \checkmark$

7. [4 marks] Write one of the vectors below as a linear combination of the other two:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 10 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}.$$

Solve $a\vec{v_1} + b\vec{v_2} + c\vec{v_3} = \vec{0}$

then rearrange

$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 3 & 1 & 2 & 0 \\ 10 & 1 & 5 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 10R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 7 & 5 & 0 \\ 0 & 21 & 15 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3 - 3R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 7 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{c = t}$

$7b + 5c = 0$ $\quad b = -\frac{5}{7}c \quad \boxed{b = \frac{-5}{7}t}$

$a - 2b - c = 0$ $\quad a = 2b + c \quad a = -\frac{10}{7}t + t$

$\boxed{a = \frac{-3}{7}t}$

t can be any nonzero value

e.g. $t = 1$

$a = \frac{-3}{7}, b = \frac{-5}{7}, c = 1$

$\frac{-3}{7}\vec{v_1} - \frac{5}{7}\vec{v_2} + \vec{v_3} = 0$

$\boxed{\vec{v_3} = \frac{3}{7}\vec{v_1} + \frac{5}{7}\vec{v_2}}$

Other possibilities as well.