1. [6 marks] Let $\mathbf{u} = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ -2 \\ 3 \end{bmatrix}$.

a) Find $||5\mathbf{u} - 4\mathbf{v}||$.

$$5\vec{u} - 4\vec{v} = 5\begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} - 4\begin{bmatrix} -6 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ -25 \\ 35 \end{bmatrix} + \begin{bmatrix} 24 \\ 8 \\ -12 \end{bmatrix}$$

$$= \begin{bmatrix} 39 \\ -17 \\ 23 \end{bmatrix}$$

$$||5\vec{u} - 4\vec{v}|| = \sqrt{39^2 + (-17)^2 + 23^2}$$

$$= \sqrt{2339}$$

b) Calculate the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \, ||\vec{v}|| \cos\theta$$

$$13 = \sqrt{83}\sqrt{49}\cos\theta$$

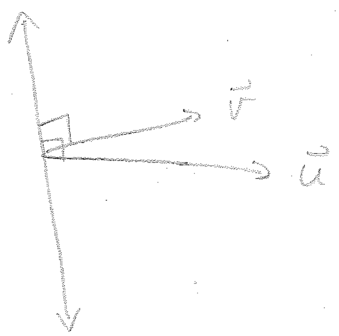
$$\cos\theta = \frac{13}{7\sqrt{83}}$$

$$\theta = \cos^{-1}\left(\frac{13}{7\sqrt{83}}\right) \approx 78^\circ$$

2. [4 marks] Find all vectors with length 19 that are perpendicular to both

$$\mathbf{u} = \begin{bmatrix} -20 \\ 9 \\ 31 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 7 \\ -4 \\ -7 \end{bmatrix}.$$

There are two possible vectors:



$$\vec{u} \times \vec{v} = [61, 77, 17]$$

$$\boxed{\begin{array}{ccccc} -20 & 9 & 31 & -20 & 9 \\ 7 & -4 & -7 & 7 & -4 \end{array}}$$

$$\|\vec{u} \times \vec{v}\| = \sqrt{9939}$$

$$\frac{1}{\sqrt{9939}} [61, 77, 17] \quad \text{has length } 1$$

$$\frac{19}{\sqrt{9939}} [61, 77, 17] \quad " \quad 19$$

$$\boxed{\pm \frac{19}{\sqrt{9939}} [61, 77, 17]}$$

3. [5 marks] Let $\mathbf{u} = [x, 2, -2]$, $\mathbf{v} = [-3, 6, 1]$ and $\mathbf{w} = [9, -2, 3]$.

a) Find the volume of the parallelepiped determined by $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$. Your answer will involve x.

$$\text{Volume} = \left| \begin{vmatrix} x & 2 & -2 \\ -3 & 6 & 1 \\ 9 & -2 & 3 \end{vmatrix} \right|$$

$$= \left| x \begin{vmatrix} 6 & 1 \\ -2 & 3 \end{vmatrix} - 2 \begin{vmatrix} -3 & 1 \\ 9 & 3 \end{vmatrix} - 2 \begin{vmatrix} -3 & 6 \\ 9 & -2 \end{vmatrix} \right|$$

$$= \left| x(20) - 2(-18) - 2(-48) \right|$$

$$= \left| 20x + 132 \right|$$

absolute value

b) For what value(s) of x do $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ lie in a common plane?

Volume (parallelepiped) $= 0$

$$|20x + 132| = 0$$

$$20x + 132 = 0$$

$$x = \frac{-132}{20} = \frac{-33}{5}$$

4. [5 marks] Solve using Gauss-Jordan Elimination:

$$\begin{aligned} 2x - 4y + 30z &= 46 \\ 3x - 5y + 39z &= 59 \\ 7x - 11y + 87z &= 131 \end{aligned}$$

$$\begin{array}{c} \begin{array}{ccc} x & \;\; y & \;\; z \end{array} \\ \left[\begin{array}{ccc|c} 2 & -4 & 30 & 46 \\ 3 & -5 & 39 & 59 \\ 7 & -11 & 87 & 131 \end{array}\right] \end{array}$$

$$\frac{R_1}{2} \quad \left[\begin{array}{ccc|c} 1 & -2 & 15 & 23 \\ 3 & -5 & 39 & 59 \\ 7 & -11 & 87 & 131 \end{array}\right]$$

$$\begin{array}{c} \\ R_2 - 3R_1 \\ R_3 - 7R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -2 & 15 & 23 \\ 0 & 1 & -6 & -10 \\ 0 & 3 & -18 & -30 \end{array}\right]$$

$$\begin{array}{c} R_1 + 2R_2 \\ \\ R_3 - 3R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -6 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$z = t$$

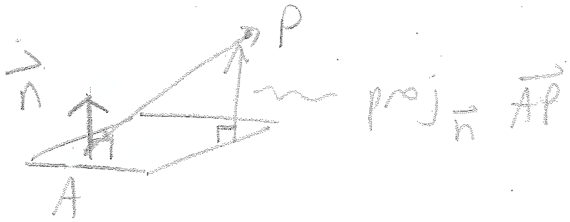
$$x + 3z = 3 \Rightarrow x = 3 - 3t$$

$$y - 6z = -10 \Rightarrow y = -10 + 6t$$

OR

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 6 \\ 1 \end{bmatrix}$$

5. [5 marks] Find the distance between the point $P = (2, -3, 7)$ and the plane $4x - 7y + 11z = 12$.



distance $= \|\text{proj}_{\vec{n}} \overrightarrow{AP}\|$

Let A be any point on the plane, say $A = (3, 0, 0)$.

$$\overrightarrow{AP} = \begin{bmatrix} -1 \\ -3 \\ 7 \end{bmatrix}$$

$$\vec{n} = \begin{bmatrix} 4 \\ -7 \\ 11 \end{bmatrix}$$

$$\text{proj}_{\vec{n}} \overrightarrow{AP} = \frac{\vec{n} \cdot \overrightarrow{AP}}{\|\vec{n}\|^2} \vec{n}$$

$$= \frac{94}{186} \begin{bmatrix} 4 \\ -7 \\ 11 \end{bmatrix}$$

$$\text{distance} = \left\| \frac{94}{186} \begin{bmatrix} 4 \\ -7 \\ 11 \end{bmatrix} \right\|$$

$$= \frac{94}{186} \left\| \begin{bmatrix} 4 \\ -7 \\ 11 \end{bmatrix} \right\|$$

$$= \frac{94\sqrt{186}}{186} \quad \text{or} \quad \frac{47\sqrt{186}}{93}$$