

## 4.1 Eigenvalues and Eigenvectors

DEF Let  $A$  be an  $n \times n$  matrix  
If  $A\vec{x} = \lambda\vec{x}$  for  $\vec{x} \neq \vec{0}$  then  
 $\lambda$  is an eigenvalue of  $A$  and  
 $\vec{x}$  is an eigenvector of  $A$

Ex: Show that  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is an eigenvector

$$\text{of } A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$$

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -4 \end{bmatrix} \\ &= 4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= 4\vec{x} \end{aligned}$$

Terminology:  $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is an eigenvector of  $A$   
corresponding to eigenvalue  $\lambda = 4$

Note:  $A\vec{0} = \lambda\vec{0}$  is trivial, so  $\vec{x} = \vec{0}$   
is not considered an eigenvector.

Ex: Find all eigenvectors of  $A = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix}$  P2  
Corresponding to  $\lambda=6$

$$\text{Want } A\bar{x} = 6\bar{x}$$

$$A\bar{x} = 6I\bar{x}$$

$$A\bar{x} - 6I\bar{x} = \bar{0}$$

$$(A - 6I)\bar{x} = \bar{0}$$

$$\text{Solve } [A - 6I | \bar{0}]$$

$$A - 6I = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix} - 6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -3 & -2 \end{bmatrix}$$

$$[A - 6I | \bar{0}]$$

$$\left[ \begin{array}{cc|c} -3 & -2 & 0 \\ -3 & -2 & 0 \end{array} \right]$$

$$\frac{R_1}{-3} \quad \left[ \begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ -3 & -2 & 0 \end{array} \right]$$

$$R_2 + 3R_1 \quad \left[ \begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\begin{matrix} \uparrow \\ x_2 = t \end{matrix}$$

$$x_1 + \frac{2}{3}x_2 = 0 \quad x_1 = -\frac{2}{3}t$$

$$\bar{x} = \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} t$$

$$\boxed{\bar{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} t \quad (t \neq 0)}$$

$$\text{CHECK: } A \begin{bmatrix} -2 \\ 3 \end{bmatrix} = 6 \begin{bmatrix} -2 \\ 3 \end{bmatrix} \checkmark$$

To find eigenvectors corresponding to eigenvalue  $\lambda$ :

$$\text{Solve } [A - \lambda I | \vec{0}]$$

Def

The eigenspace  $E_\lambda$  is the set of all eigenvectors of  $A$  corresponding to  $\lambda$ , plus the zero vector. It's a subspace of  $\mathbb{R}^n$ .

Ex: Find a basis for  $E_3$   $A = \begin{bmatrix} 4 & 1 & -2 \\ -3 & 0 & 6 \\ 2 & 2 & -1 \end{bmatrix}$

$$\text{Solve } [A - \lambda I | \vec{0}]$$

$$[A - 3I | \vec{0}]$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -3 & -3 & 6 & 0 \\ 2 & 2 & -4 & 0 \end{array} \right]$$

$$\begin{array}{l} R_2 + 3R_1 \\ R_3 - 2R_1 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\uparrow$        $\uparrow$   
 $x_2 = 1$      $x_3 = t$

$$x_1 + x_2 - 2x_3 = 0 \rightarrow x_1 = -1 + 2t$$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for  $E_3 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Ex: Find a basis for  $E_0$      $A = \begin{bmatrix} 4 & 1 & -3 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix}$  p4

Solve  $[A - I | \bar{0}]$

$[A - 0I | \bar{0}]$

$$\left[ \begin{array}{ccc|c} 4 & 1 & -3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array} \right]$$

$$\rightsquigarrow \left[ \begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\boxed{x_2 = t}$$

$$x_1 + \frac{1}{4}x_2 = 0 \rightarrow \boxed{x_1 = -\frac{1}{4}t}$$

$$\boxed{x_3 = 0}$$

$$\bar{x} = \begin{bmatrix} -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} t \quad \text{or} \quad \bar{x} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} t$$

$$\text{Basis} = \left\{ \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \right\}$$

p5

Fact

Let  $B$  be an  $n \times n$  matrix  
 $B\bar{x} = \bar{0}$  has nontrivial solutions  
 exactly when  $\det B = 0$

Fact

To find eigenvalues of  $A$ , solve  $\det(A - \lambda I) = 0$

Why?  $\lambda$  is an eigenvalue of  $A$   
 if and only if  $A\bar{x} = \lambda\bar{x}$  ( $\bar{x} \neq \bar{0}$ )  
 "  $(A - \lambda I)\bar{x} = \bar{0}$  ( $\bar{x} \neq \bar{0}$ )  
 "  $\det(A - \lambda I) = 0$

Ex: Find all eigenvalues of  $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$

Solve  $\det(A - \lambda I) = 0$

$$\textcircled{1} \quad \begin{vmatrix} 4-\lambda & -2 \\ 5 & -7-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-7-\lambda) + 10 = 0$$

$$\lambda^2 + 3\lambda - 28 + 10 = 0$$

$$\lambda^2 + 3\lambda - 18 = 0$$

$$(\lambda + 6)(\lambda - 3) = 0$$

$$\lambda = -6, 3$$

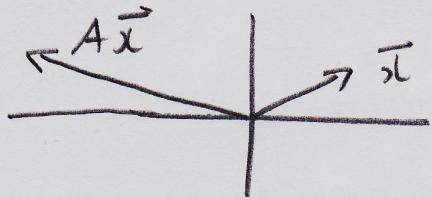
Summary

To find eigenvalues solve  $\det(A - \lambda I) = 0$   
eigenvectors  $[A - \lambda I | \vec{0}]$

$$\text{Ex: } A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

Find the eigenvectors and eigenvalues geometrically.

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ x_2 \end{bmatrix}$$



For which  $\bar{x}$  are  $\bar{x}$  and  $A\bar{x}$  parallel?

$$1) \quad \begin{array}{c} A\bar{x} \\ \leftarrow \rightarrow \\ \bar{x} \end{array} \quad \lambda = -2 \quad E_2 = \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

$$2) \quad \begin{array}{c} \uparrow \bar{x} = A\bar{x} \\ \bar{x} \end{array} \quad \lambda = 1 \quad E_1 = \text{span}(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

$$E_{-2} = \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

$$E_1 = \text{span}(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$$