3.4 LU Factorization

The LU factorization of a square matrix A:

$$A = LU$$

L: unit lower triangular (1's on main diagonal)

U: upper triangular

e.g. $\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix}$

Ex: Solve $\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$ using $A = LU$ above

$$A\vec{x} = \vec{b}$$
$$L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$$

① Let $U\vec{x} = \vec{y}$

Solve $L\vec{y} = \vec{b}$

$$\begin{array}{c} \begin{matrix} y_1 & y_2 & y_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 2 \\ 4 & 3 & 1 & -8 \end{array}\right] \end{array}$$

$2y_1 + y_2 = 2$ $\qquad$ $2 + y_2 = 2$ $\qquad$ $\boxed{y_1 = 1}$

$4y_1 + 3y_2 + y_3 = -8$ $\qquad$ $4 + y_3 = -8$ $\qquad$ $\boxed{y_2 = 0}$

$\boxed{y_3 = -12}$

(2) Solve $U\vec{x} = \vec{y}$ using $\vec{y} = \begin{bmatrix} 1 \\ 0 \\ -12 \end{bmatrix}$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 6 & -12 \end{array}\right] \end{array}$$

$6x_3 = -12$ $\qquad$ $x_3 = -2$

$2x_2 + x_3 = 0$ $\qquad$ $2x_2 - 2 = 0$ $\qquad$ $x_2 = 1$

$2x_1 + x_2 + x_3 = 1$ $\qquad$ $2x_1 - 1 = 1$ $\qquad$ $x_1 = 1$

$$\bar{x} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Finding LU

$A \rightsquigarrow REF$ using only row-k (pivot row)
Record the k-value at each step

Note: Only possible when no row swaps are required

Ex:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix}$$

$$\begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 6 & 9 \end{bmatrix} \begin{array}{l} \\ k=2 \\ k=4 \end{array}$$

$$\begin{array}{l} \\ \\ R_3 - 3R_2 \end{array} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{array}{l} \\ \\ k=3 \end{array}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

k-values in appropriate positions

$$U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

the REF

Why This Works

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary matrix for $R_2 \to R_2 - 2R_1$: $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

L undoes the operations that turn A into U

L turns U into A

$$LU = A$$

Ex: Find LU and use it to solve

$$\begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$$

(1) Find LU

$$\begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix}$$

$$\begin{array}{l} \\ R_2 - \frac{3}{2}R_1 \\ R_3 + \frac{1}{2}R_1 \end{array} \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{array}{l} \\ k = 3/2 \\ k = -1/2 \end{array}$$

$$\begin{array}{l} \\ \\ R_3 + 0R_2 \end{array} \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} k = 0$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\underbrace{L U \bar{x}}_{\bar{y}} = \bar{b}$$

(2) Solve $L\bar{y} = \bar{b}$

$$\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 3/2 & 1 & 0 & 0 \\ -1/2 & 0 & 1 & -5 \end{array}\right] \end{array}$$

$$\boxed{y_1 = 2}$$

$$\tfrac{3}{2}y_1 + y_2 = 0 \qquad 3 + y_2 = 0 \qquad \boxed{y_2 = -3}$$

$$-\tfrac{1}{2}y_1 + y_3 = -5 \qquad -1 + y_3 = -5 \qquad \boxed{y_3 = -4}$$

(3) Solve $U\bar{x} = \bar{y}$ where $\bar{y} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 2 & -4 & 0 & 2 \\ 0 & 5 & 4 & -3 \\ 0 & 0 & 2 & -4 \end{array}\right] \end{array}$$

$$2x_3 = -4 \qquad \boxed{x_3 = -2}$$

$$5x_2 + 4x_3 = -3 \qquad 5x_2 - 8 = -3 \qquad \boxed{x_2 = 1}$$

$$2x_1 - 4x_2 = 2 \qquad 2x_1 - 4 = 2 \qquad \boxed{x_1 = 3}$$

$$\bar{x} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$