

3.3 The Inverse of a Matrix

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Def

An $n \times n$ matrix A is invertible if there exists a matrix A^{-1} (also $n \times n$) so that $AA^{-1} = I$ and $A^{-1}A = I$

A^{-1} is called the inverse of A

Ex: Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ Check that $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$$AA^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$A^{-1}A = I \checkmark$$

- Notes:
- 1) Not every square matrix is invertible
 - 2) $AA^{-1} = I$ if and only if $A^{-1}A = I$
so only need to check one

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the determinant of A
is $\det A = ad - bc$

$$A^{-1} = \begin{cases} \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} & \text{if } \det A \neq 0 \\ \text{undefined} & \text{if } \det A = 0 \end{cases}$$

Ex: Find A^{-1}

a) $A = \begin{bmatrix} 1 & -4 \\ 7 & 2 \end{bmatrix}$

$$\det A = 30$$

$$A^{-1} = \frac{1}{30} \begin{bmatrix} 2 & 4 \\ -7 & 1 \end{bmatrix}$$

Check: $A^{-1}A = I$ ✓

b) $A = \begin{bmatrix} 3 & -2 \\ -9 & 6 \end{bmatrix}$

$$\det A = 0$$

A^{-1} does not exist
(A is not invertible)

System of equations: $A\bar{x} = \bar{b}$

$$\text{If } A^{-1} \text{ exists: } A^{-1}A\bar{x} = A^{-1}\bar{b}$$

$$\bar{x} = A^{-1}\bar{b}$$

$$\bar{x} = A^{-1}\bar{b}$$

FACT

If A^{-1} exists then $A\bar{x} = \bar{b}$ has the unique solution $\bar{x} = A^{-1}\bar{b}$

Ex: Use A^{-1} to solve

$$\begin{cases} 4x - 5y = -6 \\ -5x + 6y = 7 \end{cases}$$

$$A = \begin{bmatrix} 4 & -5 \\ -5 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix} = -\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}$$

$$\bar{x} = A^{-1}\bar{b}$$

$$= -\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} -6 \\ 7 \end{bmatrix}$$

$$= -\begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Finding A^{-1} for $n \times n$ matrices

Ex: $[A | I] \rightsquigarrow [I | A^{-1}]$ using row operations

Ex: Find A^{-1} for $A = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

$[A | I]$

$$\left[\begin{array}{ccc|ccc} 2 & 5 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$R_1 \leftrightarrow R_2 \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 5 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$R_2 - 2R_1 \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array} \right]$$
$$R_3 - 2R_1 \left[\begin{array}{ccc|ccc} 0 & -2 & -2 & 0 & -2 & 1 \end{array} \right]$$

$$R_1 - 2R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & -2 & 5 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & -8 & 2 & -6 & 1 \end{array} \right]$$
$$R_3 + 2R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & -2 & 5 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & 0 & 2 & -4 & 1 \end{array} \right]$$

$$R_3 / (-8) \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & -2 & 5 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{5}{8} & -\frac{1}{8} \end{array} \right]$$

$$R_1 - 8R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & \frac{2}{8} & \frac{2}{8} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{2}{8} & \frac{6}{8} & -\frac{1}{8} \end{array} \right]$$

$$R_2 + 3R_3 \quad A^{-1} = \frac{1}{8} \begin{bmatrix} 0 & -8 & 8 \\ 2 & 2 & -3 \\ -2 & 6 & -1 \end{bmatrix}$$

$$A^{-1}A = I \checkmark$$

Ex: Find A^{-1} for $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 6 \\ 2 & 3 & 11 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 6 & 0 & 1 & 0 \\ 2 & 3 & 11 & 0 & 0 & 1 \end{array} \right]$$

$$R_2 - R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array} \right]$$

$$R_3 - R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right]$$

If a zero row appears on the left side, then A^{-1} does not exist.

3 Properties of A^{-1}

① If A^{-1} exists then $(A^{-1})^{-1} = A$

② $(A^T)^{-1} = (A^{-1})^T$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

$$(A^T)^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

③ $(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \cdots A_2^{-1} A_1^{-1}$

In particular: $(AB)^{-1} = B^{-1} A^{-1}$

Ex: Operation A: put on socks
B: " shoes

To undo: undo operations and reverse order

Note: Since $(A^n)^{-1} = (A^{-1})^n$ we can write
 A^{-n} without confusion.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Find A^{-2}

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$A^{-2} = (A^2)^{-1} = \frac{1}{4} \begin{bmatrix} 22 & -10 \\ -15 & 7 \end{bmatrix}$$

Ex: A, B, X are all $n \times n$ invertible matrices
Solve for X given $(AX)^{-1} = BA$

$$((AX)^{-1})^{-1} = (BA)^{-1}$$

$$AX = (BA)^{-1}$$

$$AX = A^{-1}B^{-1}$$

Apply A^{-1} to left: $X = A^{-1}A^{-1}B^{-1}$

$$X = A^{-2}B^{-1}$$

Elementary Matrices represent row operations

How has I changed?

Ex: a) $E_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ represents $3R_1 \begin{bmatrix} & \\ & \end{bmatrix}$

$\frac{1}{3}R_1 \begin{bmatrix} & \\ & \end{bmatrix}$ under it

$$E_1^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

b) $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents $R_1 \leftrightarrow R_2 \begin{bmatrix} & \\ & \end{bmatrix}$

$R_1 \leftrightarrow R_2 \begin{bmatrix} & \\ & \end{bmatrix}$ under it

$$E_2^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

c) $E_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ represents $R_2 + 2R_1 \begin{bmatrix} & \\ & \end{bmatrix}$

$R_2 - 2R_1 \begin{bmatrix} & \\ & \end{bmatrix}$ under it

$$E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

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Elementary matrices act on the left of A

e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ c & d \end{bmatrix}$

Ex: $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

Write A and A^{-1} as a product of elementary matrices.

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$R_1/2 \quad \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

$$R_1 - \frac{1}{2}R_2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_2 E_1 A = I$$

$$A^{-1} = E_2 E_1$$

$$A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A &= (A^{-1})^{-1} \\ &= E_1^{-1} E_2^{-1} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\text{Ex: } A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

Write A and A^{-1} as a product of elementary matrices

$$R_{1/2} \quad \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$R_2 - R_1 \quad \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$R_2(-1) \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$R_1 - 2R_2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_4 E_3 E_2 E_1 A = I$$

$$A^{-1} = E_4 E_3 E_2 E_1$$

$$= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

The Fundamental Theorem of Invertible Matrices

Let A be an $n \times n$ matrix. The following statements are equivalent:

- a. A is invertible.
- b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every \mathbf{b} in \mathbb{R}^n .
- c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- d. The RREF of A is I_n .
- e. A is a product of elementary matrices.

Handout

For a given $n \times n$ matrix, the five statements are all true or all false

Ex: $A = \begin{bmatrix} 1 & 4 \\ 6 & 9 \end{bmatrix}$

A^{-1} exists

- a) A is invertible
 - b)
 - c)
 - d)
 - e)
- $\left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \text{all true}$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

A^{-1} does not exist

- a) A is invertible
 - b)
 - c)
 - d)
 - e)
- $\left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \text{all false}$