3.1 Matrix Operations

Size of a matrix : (# rows) × (# columns)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{is} \quad 2 \times 3$$

Entry of a matrix : $a_{23} = 6$ or $[A]_{23} = 6$

Square matrix : has size 2×2, 3×3 etc.

Identity matrix : $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ etc.

Diagonal matrix : $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ etc.

Ex: $A = \begin{bmatrix} 1 & 6 & 1 \\ -2 & -2 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 6 & 9 \end{bmatrix}$

Find :

a) $A + B = \begin{bmatrix} 2 & 6 & -2 \\ -1 & 4 & 13 \end{bmatrix}$

$A+B$ is undefined if A, B have differet sizes

b) $3A = \begin{bmatrix} 3 & 18 & 3 \\ -6 & -6 & 12 \end{bmatrix}$

"scalar multiplication"

c) $A - 3B$

$= A + (-3B)$

$= \begin{bmatrix} 1 & 6 & 1 \\ -2 & -2 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 9 \\ -3 & -18 & -27 \end{bmatrix}$

$= \begin{bmatrix} -2 & 6 & 10 \\ -5 & -20 & -23 \end{bmatrix}$

Def The <u>transpose</u> of A, written A^T, has rows and columns interchanged

A is <u>symmetric</u> if $A^T = A$

<u>Ex</u>: a) $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 6 & 3 \\ 4 & 3 & -1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 6 & 3 \\ 4 & 3 & -1 \end{bmatrix}$

A is symmetric

b) $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix}$

$B^T = \begin{bmatrix} 1 & 0 \\ 2 & 6 \\ 1 & 1 \end{bmatrix}$

B is not symmetric

Matrix Multiplication

$$AB = \begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 \\ r_2 \cdot c_1 & \cdots \end{bmatrix}$$

where $r_i = i^{th}$ row of $\underline{A}$

$c_j = j^{th}$ column of $\underline{B}$

<u>Ex:</u>

$$\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 9 & 27 \\ -2 & 0 & 0 \end{bmatrix}$$

$$[1 \ \ 4] \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 1(3) + 4(6) = 27$$

Consider the sizes:

$$(2 \times \underbrace{2)(2}_{} \times 3) = (2 \times 3)$$

- Inner numbers must be equal, otherwise AB is undefined
- Outer numbers give size of AB

<u>Ex:</u> A is 2×3 and B is 3×1

AB is 2×1

BA is undefined

$\boxed{\text{Fact: } AB \neq BA \text{ in general}}$

Ex: Find BC and CB

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$$BC = \begin{bmatrix} 5 & 7 \\ 11 & 13 \\ 17 & 19 \end{bmatrix}$$

CB is undefined

Why do we multiply like this?

Consider $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$\begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{cases} x + 2y = 5 \\ 3x + 4y = 6 \end{cases}$$

Fact: A system of equations can be written $A\bar{x} = \bar{b}$

Coefficients (A), variables (column) ($\bar{x}$), constants (column) ($\bar{b}$)

Matrix multiplication is designed to solve systems.

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Ex: A: test marks

$$\begin{array}{c} \\ T1 \\ T2 \\ Exam \end{array} \begin{array}{c} \begin{array}{cc} Al & Bob \end{array} \\ \begin{bmatrix} 50 & 60 \\ 90 & 80 \\ 75 & 70 \end{bmatrix} \end{array}$$

B: weightings

$$\begin{array}{c} \begin{array}{ccc} T1 & T2 & Exam \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.6 \end{bmatrix} \end{array}$$

Find Al and Bob's final grades

→ Need compatible sizes and categories

$$BA = \begin{array}{c} \begin{array}{ccc} T1 & T2 & Exam \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.6 \end{bmatrix} \end{array} \begin{array}{c} \\ T1 \\ T2 \\ Exam \end{array} \begin{array}{c} \begin{array}{cc} Al & Bob \end{array} \\ \begin{bmatrix} 50 & 60 \\ 90 & 80 \\ 75 & 70 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{cc} Al & Bob \end{array} \\ \begin{bmatrix} 73 & 70 \end{bmatrix} \end{array}$$

Ex: Find 2nd column of

$$\begin{bmatrix} 1 & 2 \\ 2 & 6 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 4 & 7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ 46 \\ 65 \end{bmatrix}$$

Notice $\begin{bmatrix} 16 \\ 46 \\ 65 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$

Fact: Columns of AB are linear combinations of the columns of A

Will be useful in section 7.3

Outer product expansion of AB

$$[A_1 | A_2 | \cdots | A_n] \begin{bmatrix} B_1 \\ \hline B_2 \\ \hline \vdots \\ \hline B_n \end{bmatrix}$$

$$= A_1B_1 + A_2B_2 + \cdots + A_nB_n$$

Ex: $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 7 \\ 4 & 2 \end{bmatrix}$

Find the outer product expansion of AB

$AB = A_1B_1 + A_2B_2$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 7 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}\begin{bmatrix} 4 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 7 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 12 & 6 \\ -8 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 13 \\ -8 & -4 \end{bmatrix}$$

(Useful in section 5.4)

Powers of a Matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$A^2 = AA$$
$$= \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}$$

Usual exponent rules apply

e.g. $A^3 = A^2 A$ or $A^3 = AA^2$

$A^6 = (A^3)^2$

Fact: $AI = A$ and $IA = A$ for any matrix A

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Ex: Simplify B^{2018} given that $B^3 = I$

$2018 = 3(?) + ?$

$\frac{2018}{3} \approx 672.7$

$2018 = 3(672) + ?$

$2018 = 3(672) + 2$

$$\begin{aligned} B^{2018} &= B^{3(672)+2} \\ &= B^{3(672)} \cdot B^2 \\ &= (B^3)^{672} \cdot B^2 \\ &= I^{672} \cdot B^2 \\ &= I \cdot B^2 \\ &= B^2 \end{aligned}$$

Def

Let $\mathbb{O}$ be the zero matrix

e.g. $\mathbb{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\mathbb{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ etc.

Ex: Find a 2x2 matrix A so that

$A^2 = \mathbb{O}$ but $A \neq \mathbb{O}$

Many possible answers $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$