

(1.4) Cross Product ~~WEEK 11~~ ~~WEEK 12~~

Cross Product $\bar{u} \times \bar{v}$ is defined for \bar{u}, \bar{v} in \mathbb{R}^3

Ex: $\bar{u} = [1, 2, 1]$ $\bar{v} = [3, -1, 4]$

$$\begin{array}{r} 1 \\ 2 \\ 3 \end{array} \times \begin{array}{r} 3 \\ -1 \\ 4 \end{array} = \begin{array}{r} 1 \\ 2 \\ 3 \end{array} \times \begin{array}{r} 1 \\ 1 \\ 3 \end{array} \times \begin{array}{r} 2 \\ -1 \\ -1 \end{array}$$

$$\begin{aligned} \bar{u} \times \bar{v} &= [2(4) - 1(-1), 1(3) - 1(4), 1(-1) - 2(3)] \\ &= [9, -1, -7] \end{aligned}$$

Ex: a) Compute $\bar{v} \times \bar{u}$

$$\begin{array}{r} 3 \\ -1 \\ 1 \end{array} \times \begin{array}{r} 4 \\ 1 \\ 2 \end{array} = \begin{array}{r} 3 \\ 1 \\ 1 \end{array} \times \begin{array}{r} -1 \\ 2 \\ 2 \end{array}$$

$$\bar{v} \times \bar{u} = [-9, 1, 7]$$

b) Compute $(\bar{u} \times \bar{v}) \cdot \bar{u}$

$$= [9, -1, -7] \cdot [1, 2, 1]$$

$$= 9 - 2 - 7$$

$$= 0$$

FACTS

$$\bar{v} \times \bar{u} = -(\bar{u} \times \bar{v})$$

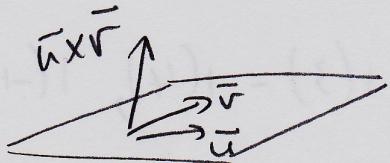
$\bar{u} \times \bar{v}$ is orthogonal to both \bar{u} and \bar{v}

FACT

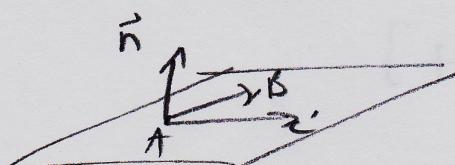
p2

$\bar{u} \times \bar{v}$ is a normal for the plane containing \bar{u} and \bar{v}

Direction of $\bar{u} \times \bar{v}$ is determined by the Right Hand Rule



Ex: Find the general form of the plane through $A(1, 3, 6)$, $B(2, 1, 4)$ and $C(1, -1, 5)$.



$$\bar{AB} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

$$\bar{AC} = \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}$$

$$\bar{n} = \bar{AB} \times \bar{AC} = [-6, 1, -4] \quad \begin{matrix} 1 & -2 & 2 \\ 0 & -4 & -1 \\ 0 & 0 & -4 \end{matrix}$$

$$\bar{n} \cdot \bar{x} = \bar{n} \cdot \bar{p}$$

$$\begin{bmatrix} -6 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

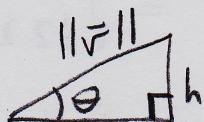
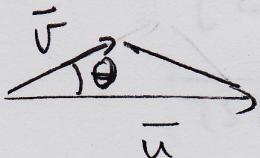
$$-6x + y - 4z = -27$$

$$\text{or } 6x - y + 4z = 27 \quad \text{etc.}$$

Recall $\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos \theta$

FACT $\|\bar{u} \times \bar{v}\| = \|\bar{u}\| \|\bar{v}\| \sin \theta$
for \bar{u}, \bar{v} in \mathbb{R}^3

Ex: Show that the area of the triangle below is $\frac{1}{2} \|\bar{u} \times \bar{v}\|$



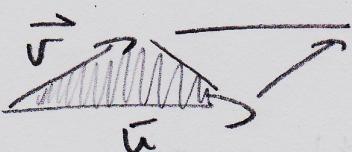
$$\sin \theta = \frac{h}{\|\bar{v}\|}$$

$$h = \|\bar{v}\| \sin \theta$$

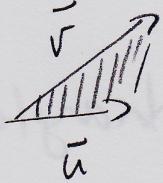
$$\begin{aligned} \text{Area } (\triangle) &= \frac{1}{2} \text{ base} \cdot \text{height} \\ &= \frac{1}{2} \|\bar{u}\| \|\bar{v}\| \sin \theta \\ &= \frac{1}{2} \|\bar{u} \times \bar{v}\| \end{aligned}$$

FACT $A(\text{triangle}) = \frac{1}{2} \|\bar{u} \times \bar{v}\|$

$A(\text{parallelogram}) = \|\bar{u} \times \bar{v}\|$
in \mathbb{R}^3



Ex: Find the area of the triangle determined by $\bar{u} = [1, 4, 5]$ and $\bar{v} = [2, 3, 6]$



$$\begin{matrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{matrix} \times \begin{matrix} 1 & 4 \\ 2 & 3 \end{matrix}$$

$$\bar{u} \times \bar{v} = [9, 4, -5]$$

$$\|\bar{u} \times \bar{v}\| = \sqrt{81 + 16 + 25} \\ = \sqrt{122}$$

$$\text{Area}(\Delta) = \frac{1}{2} \|\bar{u} \times \bar{v}\| \\ = \frac{\sqrt{122}}{2}$$

Matrix: rectangular array

$$\text{e.g. } A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

Size : (# rows) x (# columns)

e.g. A is 2×3

The determinant of A is written $\det A$ or $|A|$.
Only defined for square matrices.

FORMULAS

$$1) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ or } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$2) \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a |ef| - b |dg| + c |gh|$$

↑
Caution

"Cofactor expansion"

Ex: Compute

a) $\det \begin{bmatrix} 1 & 4 & 6 \\ 2 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$

$$= 1 \begin{vmatrix} 1 & 3 \\ 6 & 7 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 \\ 0 & 7 \end{vmatrix} + 6 \begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix}$$

$$= 1(-11) - 4(14) + 6(12)$$

$$= 5$$

b) $\begin{vmatrix} -1 & -4 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 8 \end{vmatrix}$

$$= -1(6) + 4(6) + 6(0)$$

$$= 18$$

NOTATION

Let $\bar{i} = [1, 0, 0]$
 $\bar{j} = [0, 1, 0]$
 $\bar{k} = [0, 0, 1]$

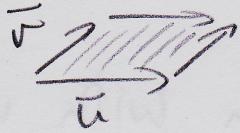
2nd Method for Cross Product

$$\begin{aligned} [2, 1, 3] \times [-6, 4, 2] &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & 3 \\ -6 & 4 & 2 \end{vmatrix} \\ &= \bar{i}(-10) - \bar{j}(22) + \bar{k}(14) \\ &= -10[1, 0, 0] - 22[0, 1, 0] + 14[0, 0, 1] \\ &= [-10, -22, 14] \end{aligned}$$

3 Geometry Formulas

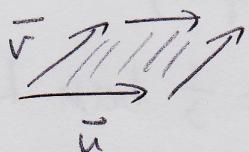
p7

1) $A(\text{parallelogram in } \mathbb{R}^3) = \|\bar{u} \times \bar{v}\|$



2) $A(\text{parallelogram in } \mathbb{R}^2)$

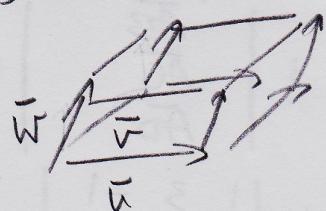
= absolute value of $\det \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix}$



3) $V(\text{parallelepiped in } \mathbb{R}^3)$

slanted box

= absolute value of $\det \begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{bmatrix}$



Ex: Do vectors $[1, 4, 7]$, $[2, 5, 9]$, $[1, -2, -3]$ lie in a plane?

Yes if and only if $V(\text{parallelepiped}) = 0$

$$V(\text{parallelepiped}) = 1 \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 9 \\ 1 & -2 & -3 \end{vmatrix}_1 = 1 |(3) - 4(-15) + 7(-1)| \\ = 10 \\ = 0$$

Yes

Ex: Area of parallelogram
determined by $[1, 6]$ and $[3, 5]$?

$$\left| \begin{vmatrix} 1 & 6 \\ 3 & 5 \end{vmatrix} \right|_1 = | -13 | = 13$$