1. [6 marks] $T = 0.02x^2y - 0.03xy^2$ gives temperature (in °C) in a small flat town. The variables x and y represent position (in km).

a) A runner travels from $(x, y) = (4, -3)$ to $(x, y) = (7, 2)$. What initial rate of change of temperature does the runner experience?

direction $= [3, 5]$

$\vec{u} = \frac{1}{\sqrt{34}}[3, 5]$

$\nabla T = [0.04xy - 0.03y^2,\ 0.02x^2 - 0.06xy]$

$\nabla T(4,-3) = [-0.75, 1.04]$

$D_{\vec{u}} T = \nabla T \cdot \vec{u}$

$= \frac{1}{\sqrt{34}}[-0.75, 1.04] \cdot [3, 5]$

$= \frac{2.95}{\sqrt{34}}$

$\approx 0.51 \ \frac{°C}{km}$

b) Starting from $(x, y) = (4, -3)$, in which direction does the temperature increase fastest?

$[-0.75, 1.04]$ or any positive multiple of this

c) Starting from $(x, y) = (4, -3)$, what is the maximum rate of change of temperature the runner could experience?

$\|[-0.75, 1.04]\| \approx 1.28 \ \frac{°C}{km}$

2. [4 marks] Evaluate: $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \int_{4}^{5+\sin\theta} r \, dr \, d\theta$

$$= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} r^2 \Big|_{r=4}^{r=5+\sin\theta} d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \left[(5+\sin\theta)^2 - 16 \right] d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \left(25 + 10\sin\theta + \underbrace{\sin^2\theta}_{\frac{1}{2} - \frac{\cos 2\theta}{2}} - 16 \right) d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \left(\frac{19}{2} + 10\sin\theta - \frac{\cos 2\theta}{2} \right) d\theta$$

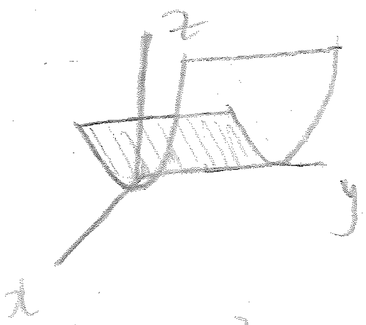
$$= \frac{1}{2} \left[\frac{19}{2}\theta - 10\cos\theta - \frac{\sin 2\theta}{4} \right]_{\frac{\pi}{2}}^{\pi}$$

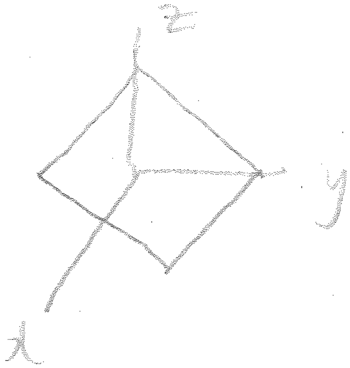
$$= \frac{1}{2} \left[\frac{19\pi}{2} + 10 - \left(\frac{19\pi}{4} \right) \right]$$

$$= \frac{1}{2} \left[\frac{19\pi}{4} + 10 \right]$$

$$= \frac{19\pi}{8} + 5$$

3. [4 marks] Set up a triple integral for the volume of the solid bounded by $z = x^2$, $y + z = 25$, and $y = 0$. Do not evaluate the integral.





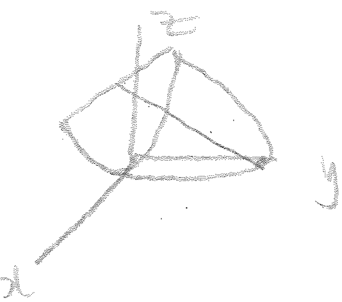
$z = x^2$

$y + z = 25$

$z = 25 - y$

$\boxed{y = 0 \text{ is the } xz\text{-plane}}$

The solid :



Slice in the z-direction:

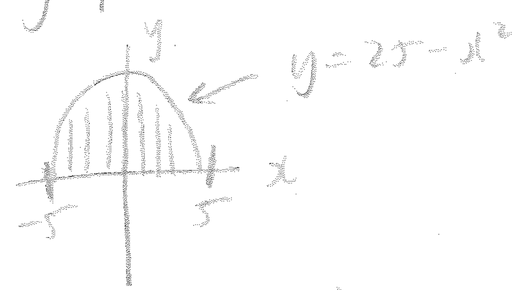
$$x^2 \le z \le 25 - y$$

Project on the xy-plane:

$z = z$

$x^2 = 25 - y$

$y = 25 - x^2$



R: $0 \le y \le 25 - x^2$

$-5 \le x \le 5$

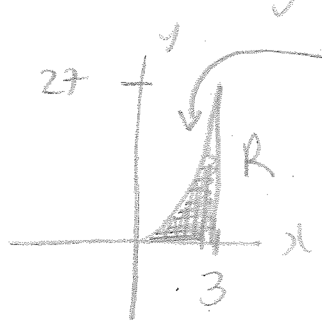
$$V = \int_{-5}^{5} \int_{0}^{25-x^2} \int_{x^2}^{25-y} dz\, dy\, dx$$

4. [3 marks] Rewrite the integral using vertical slices instead of horizontal slices. Do not evaluate.

$$\int_0^{27} \int_{\sqrt[3]{y}}^{3} \sqrt{1+x^4}\, dx\, dy$$

R: $\sqrt[3]{y} \le x \le 3$

$0 \le y \le 27$



$x = \sqrt[3]{y}$

$x^3 = y$

$y = x^3$

R: $0 \le y \le x^3$

$0 \le x \le 3$

Integral $= \displaystyle\int_0^3 \int_0^{x^3} \sqrt{1+x^4}\, dy\, dx$

5. [6 marks] Given $\overbrace{2x+5y+3z}^{g} = 26$. Use the Lagrange Multiplier method to find the point (x, y, z) at which $f = (x-3)^2+(y+1)^2+(z-2)^2$ is minimized.

$$\nabla f = \lambda \nabla g$$

$$[\,2(x-3),\ 2(y+1),\ 2(z-2)\,] = \lambda\,[2, 5, 3]$$

$$2(x-3) = 2\lambda \quad \Rightarrow \quad \lambda = x-3$$

$$2(y+1) = 5\lambda \quad \Rightarrow \quad \lambda = \frac{2}{5}(y+1)$$

$$2(z-2) = 3\lambda \quad \Rightarrow \quad \lambda = \frac{2}{3}(z-2)$$

$\lambda = \lambda = \lambda$

Solve for both y and z in terms of x

$$x-3 = \frac{2}{5}(y+1) = \frac{2}{3}(z-2)$$

$$x-3 = \frac{2}{5}(y+1)$$

$$\frac{5}{2}(x-3) = y+1$$

$$y = \frac{5}{2}(x-3)-1$$

$$y = \frac{5}{2}x - \frac{17}{2}$$

$$x-3 = \frac{2}{3}(z-2)$$

$$\frac{3}{2}(x-3) = z-2$$

$$z = \frac{3}{2}(x-3)+2$$

$$z = \frac{3}{2}x - \frac{5}{2}$$

$$y = \frac{5}{2}x - \frac{17}{2}$$

$$z = \frac{3}{2}x - \frac{5}{2}$$

$$\rightarrow \quad 2x+5y+3z = 26$$

$$2x + \frac{25}{2}x - \frac{85}{2} + \frac{9}{2}x - \frac{15}{2} = 26$$

$$19x = 76$$

$$x = 4$$

$$\Rightarrow y = \frac{5}{2}x - \frac{17}{2} = \frac{3}{2}$$

$$\text{and } z = \frac{3}{2}x - \frac{5}{2} = \frac{7}{2}$$

f is minimized at $(x, y, z) = \left(4, \frac{3}{2}, \frac{7}{2}\right)$