1. [5 marks] Find all the critical points for the following function.
Give your answer(s) in the form (x, y).
$z = 2x^2 - x^2y + 6y^2 + y^3$

$$\left.\begin{aligned} Z_x &= 4x - 2xy \\ Z_y &= -x^2 + 12y + 3y^2 \end{aligned}\right\} \text{ both 0 or undefined}$$

① $4x - 2xy = 0$

② $-x^2 + 12y + 3y^2 = 0$

①: $2x(2-y) = 0$

$x = 0$ or $y = 2$ (2 Cases)

Case 1 : $x=0$

$x=0 \rightarrow$ ②: $12y + 3y^2 = 0$

$3y(4+y) = 0$

$y = 0, -4$

Critical points are $(0,0)$, $(0,-4)$

Case 2 : $y=2$

$y=2 \rightarrow$ ②: $-x^2 + 36 = 0$

$36 = x^2$

$x = \pm 6$

Critical points are $(-6,2)$, $(6,2)$

$$\boxed{(0,0),\ (0,-4),\ (-6,2),\ (6,2)}$$

2. [4 marks] Find $\frac{\partial z}{\partial y}$ given $x^3 + z^6 - xy^2 = 9y^3z + 7$.

Take $\frac{\partial}{\partial y}$:

$$6z^5 \frac{\partial z}{\partial y} - 2xy = \underbrace{27y^2z + 9y^3\frac{\partial z}{\partial y}}_{\text{Chain Rule}}$$

$$6z^5 \frac{\partial z}{\partial y} - 9y^3 \frac{\partial z}{\partial y} = 27y^2z + 2xy$$

$$\left[6z^5 - 9y^3\right]\frac{\partial z}{\partial y} = 27y^2z + 2xy$$

$$\frac{\partial z}{\partial y} = \frac{27y^2z + 2xy}{6z^5 - 9y^3}$$

3. [4 marks] Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ given:

$f = 2\ln(x^2 - y^3) + 3e^{xy} + 6\sin(x^4y^4) - 7\cos(2x - 3y)$

$$\frac{\partial f}{\partial x} = 2\,\frac{1}{x^2-y^3}(2x) + 3ye^{xy} + 24x^3y^4\cos(x^4y^4) + 14\sin(2x-3y)$$

$$= \frac{4x}{x^2-y^3} + 3ye^{xy} + 24x^3y^4\cos x^4y^4 + 14\sin(2x-3y)$$

$$\frac{\partial f}{\partial y} = 2\,\frac{1}{x^2-y^3}(-3y^2) + 3xe^{xy} + 24x^4y^3\cos(x^4y^4) - 21\sin(2x-3y)$$

$$= \frac{-6y^2}{x^2-y^3} + 3xe^{xy} + 24x^4y^3\cos x^4y^4 - 21\sin(2x-3y)$$

4. [5 marks] Find the equation of the tangent plane to $z = \sqrt{x^2 + y^2 - 13}$ at the point on the surface where $x = 5$ and $y = -2$.

$$z_x = \frac{1}{2}(x^2+y^2-13)^{-1/2}(2x)$$

$$= \frac{x}{\sqrt{x^2+y^2-13}}$$

$$z_y = \frac{y}{\sqrt{x^2+y^2-13}}$$

$$\vec{n} = [-z_x, -z_y, 1]$$

$$= \left[\frac{-x}{\sqrt{x^2+y^2-13}}, \frac{-y}{\sqrt{x^2+y^2-13}}, 1\right]$$

$$= \left[\frac{-5}{4} \quad \frac{2}{4}, 1\right]$$

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

$$\begin{bmatrix} -5/4 \\ 2/4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5/4 \\ 2/4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

$z = \sqrt{x^2+y^2-13}$

or $$-\frac{5}{4}x + \frac{2}{4}y + z = \frac{-13}{4}$$

or $$-5x + 2y + 4z = -13$$

5. [5 marks] Let $f = \dfrac{6(\sqrt{y})}{x^5}$.
The maximum relative error in x is $\pm 3\%$.
The maximum relative error in y is $\pm 8\%$.
Estimate the maximum relative error in f.

$$f = 6 y^{1/2} x^{-5}$$

$$f_x = -30 y^{1/2} x^{-6} = \frac{-30\sqrt{y}}{x^6}$$

$$f_y = 3 y^{-1/2} x^{-5} = \frac{3}{\sqrt{y}\, x^5}$$

$$df = f_x\, dx + f_y\, dy$$

$$= \frac{-30\sqrt{y}}{x^6}\, dx + \frac{3}{\sqrt{y}\, x^5}\, dy$$

$$\frac{df}{f} = \frac{-30\sqrt{y}}{x^6}\left(\frac{x^5}{6\sqrt{y}}\right) dx + \frac{3}{\sqrt{y}\, x^5}\left(\frac{x^5}{6\sqrt{y}}\right) dy$$

$$= -5\,\frac{dx}{x} + \frac{1}{2}\,\frac{dy}{y}$$

To maximize this, use $\frac{dx}{x} = -3\%$ and $\frac{dy}{y} = 8\%$.

$$\left(\frac{df}{f}\right)_{max} = -5(-3\%) + \frac{1}{2}(8\%)$$

$$= 19\%$$