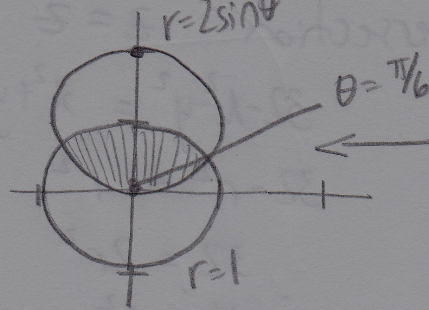
(14)



Intersection $\quad r=r$

$$1=2\sin\theta$$
$$\tfrac{1}{2}=\sin\theta$$
$$\theta=\frac{\pi}{6},\ \frac{5\pi}{6}$$

Use symmetry

$$A=2\left[\int_0^{\pi/6}\int_0^{2\sin\theta} r\,dr\,d\theta+\int_{\pi/6}^{\pi/2}\int_0^{1} r\,dr\,d\theta\right]$$

$$=2\left[\int_0^{\pi/6}\frac{1}{2}(2\sin\theta)^2\,d\theta+\int_{\pi/6}^{\pi/2}\frac{1}{2}\,d\theta\right]$$

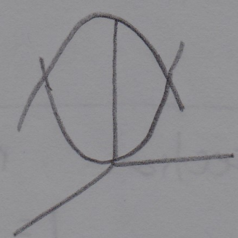
$$=2\left[\int_0^{\pi/6}\underbrace{\cancel{2\sin^2\theta}}_{1-\cos 2\theta}\,d\theta+\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\right]$$

$$=2\left[\left(\theta-\frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/6}+\frac{\pi}{6}\right]$$

$$=2\left[\frac{\pi}{6}-\frac{\sqrt{3}}{4}+\frac{\pi}{6}\right]$$

$$=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$

(15) a)



$\leftarrow$ Intersection $z = z$

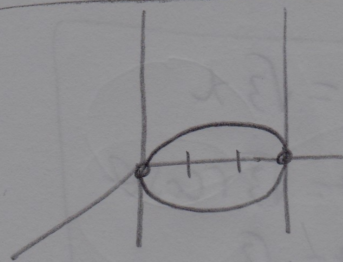
$$32 - x^2 - y^2 = x^2 + y^2$$
$$32 - r^2 = r^2$$
$$32 = 2r^2$$
$$16 = r^2$$
$$r = 4$$

$$0 \leq r \leq 4$$
$$0 \leq \theta \leq 2\pi$$
$$dA = r\,dr\,d\theta$$

$$V = \int_0^{2\pi} \int_0^4 (32 - r^2 - r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^4 (32r - 2r^3)\,dr\,d\theta$$

15 b)

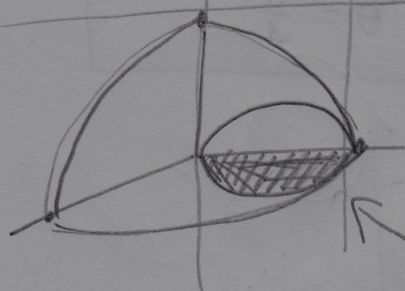


$x^2+y^2=3y$

$r^2 = 3r\sin\theta$

$r = 3\sin\theta$

$$\boxed{\begin{aligned} &0 \le r \le 3\sin\theta \\ &0 \le \theta \le \tfrac{\pi}{2} \end{aligned}}$$

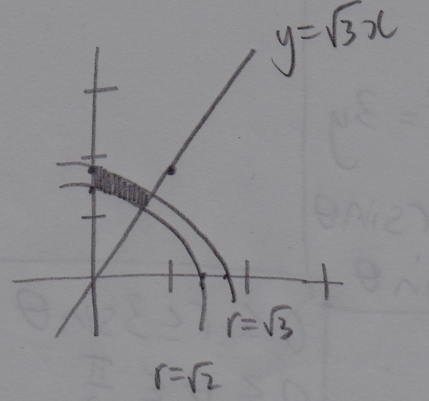


$dA = r\,dr\,d\theta$

$$\boxed{\begin{aligned} &z^2 = 9-x^2-y^2 \\ &z = \sqrt{9-r^2} \end{aligned}}$$

$$V = \int_0^{\pi/2}\int_0^{3\sin\theta} \sqrt{9-r^2}\; r\,dr\,d\theta$$

(16)



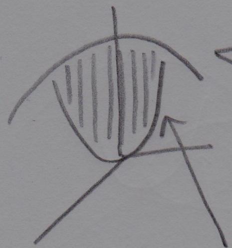
$$\boxed{\begin{gathered} y=\sqrt{3}x \\ r\sin\theta=\sqrt{3}r\cos\theta \\ \tan\theta=\sqrt{3} \\ \theta=\frac{\pi}{3} \end{gathered}}$$

$$\boxed{\begin{gathered} \sqrt{2}\le r\le\sqrt{3} \\ \frac{\pi}{3}\le\theta\le\frac{\pi}{2} \\ dA=r\,dr\,d\theta \end{gathered}}$$

$$\delta=y=r\sin\theta$$

$$m=\int_{\pi/3}^{\pi/2}\int_{\sqrt{2}}^{\sqrt{3}} r^2\sin\theta\,dr\,d\theta$$

(17)



$z^2 = 12 - x^2 - y^2$

$z = \sqrt{12-x^2-y^2}$

Intersection:

$$\underbrace{x^2+y^2}_{z} + z^2 = 12$$

$$z + z^2 = 12$$

$$z^2 + z - 12 = 0$$

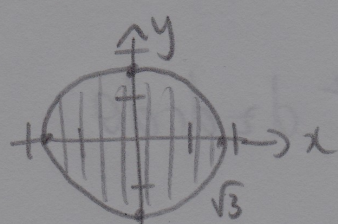
$$(z+4)(z-3) = 0$$

$$z = -4, 3$$

$$\Rightarrow z = 3$$

$$x^2+y^2 = 3$$

$$y = \pm\sqrt{3-x^2}$$



$$\boxed{\begin{array}{c} -\sqrt{3-x^2} \le y \le \sqrt{3-x^2} \\ -\sqrt{3} \le x \le \sqrt{3} \end{array}}$$

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{x^2+y^2}^{\sqrt{12-x^2-y^2}} dz\, dy\, dx$$

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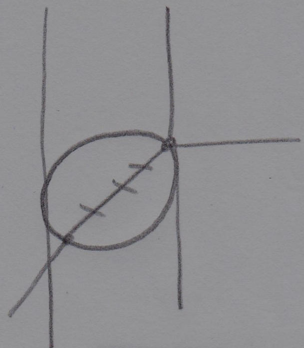
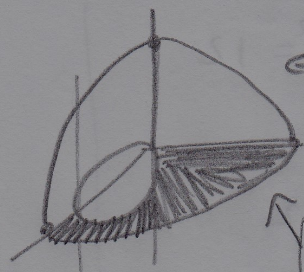


$x^2+y^2=4x$

$r^2=4r\cos\theta$

$r=4\cos\theta$



$\leftarrow\ z=\sqrt{25-r^2}$

at $z=0$: $\sqrt{25-x^2-y^2}=0$

$x^2+y^2=25$

$r=5$

$0\le z\le\sqrt{25-r^2}$

$4\cos\theta\le r\le 5$

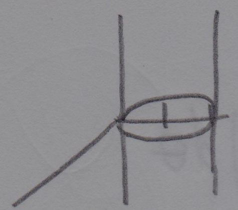
$0\le\theta\le\frac{\pi}{2}$

$dV=r\,dz\,dr\,d\theta$

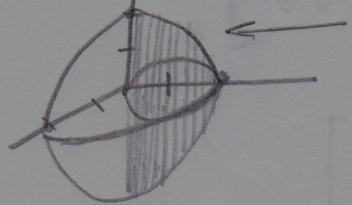
$\sqrt{x^2+y^2}=r$

$$\text{Integral}=\int_0^{\pi/2}\int_{4\cos\theta}^{5}\int_0^{\sqrt{25-r^2}} r^2\,dz\,dr\,d\theta$$

(19)



$$x^2+y^2=2y$$
$$r^2=2r\sin\theta$$
$$r=2\sin\theta$$



$$z^2=4-x^2-y^2$$
$$z=\pm\sqrt{4-r^2}$$

$$\boxed{\begin{array}{l}0\le r\le 2\sin\theta\\ 0\le\theta\le\pi\end{array}}$$

$$V=\int_0^{\pi}\int_0^{2\sin\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

$$=4\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta \text{ (by symmetry)}$$

$$=4\int_0^{\pi/2}\int_0^{2\sin\theta} r\sqrt{4-r^2}\,dr\,d\theta$$

$$=4\int_0^{\pi/2}\left.-\frac{1}{2}\cdot\frac{2}{3}(4-r^2)^{3/2}\right|_{r=0}^{r=2\sin\theta}d\theta$$

$$=-\frac{4}{3}\int_0^{\pi/2}(8\cos^3\theta-8)\,d\theta$$

$\longrightarrow$

(19) Cont'd

$$= \frac{-4}{3}\int_0^{\pi/2}\left[8(1-\sin^2\theta)\cos\theta - 8\right]d\theta$$

$$= \frac{-4}{3}\left[8\left(\sin\theta - \frac{\sin^3\theta}{3}\right) - 8\theta\right]_0^{\pi/2}$$

$$= \frac{-4}{3}\left[8\left(1 - \frac{1}{3}\right) - 4\pi\right]$$

$$= \frac{-4}{3}\left[\frac{16}{3} - 4\pi\right]$$

$$= \frac{-16}{3}\left[\frac{4}{3} - \pi\right]$$

$$= \frac{-16}{9}\left[4 - 3\pi\right]$$

$$= \frac{16}{9}(3\pi - 4)$$

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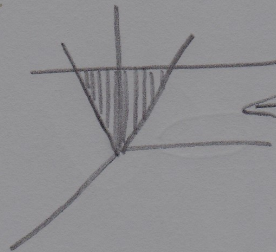


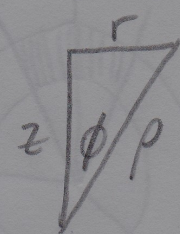
$$z = \sqrt{x^2+y^2}$$

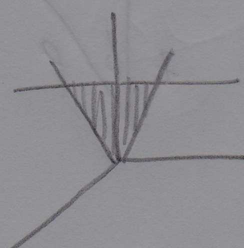
$$z = r$$

$$\rho\cos\phi = \rho\sin\phi$$

$$1 = \tan\phi$$

$$\phi = \frac{\pi}{4}$$





$$z = 2$$

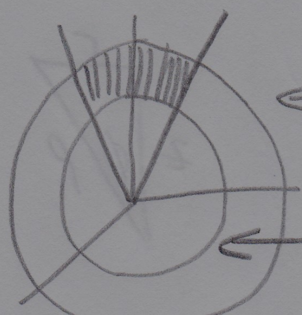
$$\rho\cos\phi = 2$$

$$\rho = 2\sec\phi$$

$$\boxed{\begin{array}{l} 0 \le \rho \le 2\sec\phi \\ 0 \le \phi \le \frac{\pi}{4} \\ 0 \le \theta \le 2\pi \\ dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \end{array}}$$

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(21)



$\rho^2 = 16$
$\rho = 4$

$\rho = 2$

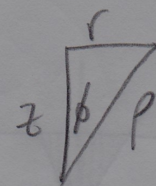
Cone: $z = \sqrt{9}\sqrt{x^2+y^2}$

$z = 3r$

$\rho\cos\phi = 3\rho\sin\phi$

$\frac{1}{3} = \tan\phi$

$\phi = \arctan\frac{1}{3}$



$2 \le \rho \le 4$

$0 \le \phi \le \arctan\frac{1}{3}$

$0 \le \theta \le 2\pi$

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

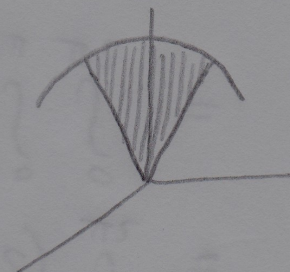
$\sqrt{x^2+y^2} = r = \rho\sin\phi$

$$\text{Integral} = \int_0^{2\pi}\int_0^{\arctan\frac{1}{3}}\int_2^4 \rho^3 \sin^2\phi \, d\rho \, d\phi \, d\theta$$

(22) $\rightarrow$ Spherical $\quad x^2+y^2+z^2 = \rho^2$

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$\sqrt{x^2+y^2} \le z \le \sqrt{8-x^2-y^2} \Rightarrow$



Intersection:

$$z = z$$
$$\sqrt{x^2+y^2} = \sqrt{8-x^2-y^2}$$
$$x^2+y^2 = 8-x^2-y^2$$
$$2(x^2+y^2) = 8$$
$$x^2+y^2 = 4$$

So $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$-2 \le x \le 2$ gives the whole cone

In spherical:

$$z = \sqrt{x^2+y^2}$$
$$z = r$$
$$\rho\cos\phi = \rho\sin\phi$$
$$1 = \tan\phi$$
$$\phi = \frac{\pi}{4}$$

$$z = \sqrt{8-x^2-y^2}$$
$$x^2+y^2+z^2 = 8$$
$$\rho^2 = 8$$
$$\rho = 2\sqrt{2}$$

$0 \le \rho \le 2\sqrt{2}$

$0 \le \phi \le \frac{\pi}{4}$

$0 \le \theta \le 2\pi$

$\rightarrow$

(22) Cont'd

$$\text{Integral} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}} \rho^4 \sin\phi \, d\rho\, d\phi\, d\theta$$

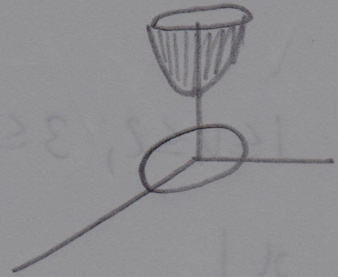
$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{(2\sqrt{2})^5}{5} \sin\phi \, d\phi\, d\theta$$

$$= \int_0^{2\pi} \frac{(2\sqrt{2})^5}{5} \left(-\cos\phi\right)\Big|_0^{\pi/4} d\theta$$

$$= 2\pi \cdot \frac{(2\sqrt{2})^5}{5} \left(-\frac{1}{\sqrt{2}} + 1\right)$$

$$= \frac{256\pi(\sqrt{2}-1)}{5}$$

23



$z_x = 2x$

$z_y = 2y$

$$dS = \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA$$

$$= \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

$$SA = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

$\boxed{\to \text{Polar}}$

$$= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{8} \cdot \frac{2}{3} \left(1 + 4r^2\right)^{3/2} \Big|_{r=0}^{r=1} d\theta$$

$$= 2\pi \cdot \left(\frac{1}{12}\right)\left(5^{3/2} - 1\right)$$

$$= \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

(24) Let $u = xy$
$v = y$ so $1 \le u \le 2,\ 3 \le v \le 4$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = y = v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{v}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{v} \text{ since } v>0 \text{ on this region}$$

$$\text{Integral} = \iint_R v^2 e^u \frac{1}{v}\, du\,dv$$

$$= \int_3^4 \int_1^2 v e^u \, du\,dv$$

$$= \int_3^4 v e^u \Big|_{u=1}^{u=2} dv$$

$$= \int_3^4 v(e^2 - e)\, dv$$

$$= \frac{v^2}{2}\Big|_3^4 (e^2 - e) = \frac{7}{2}(e^2 - e)$$