① $z_x = \frac{y^3}{x}$ $z_y = 3y^2 \ln x$

At $(e, 2, 8)$:

$z_x = \frac{8}{e}$ $z_y = 12$

$\bar{n} = [-z_x, -z_y, 1]$

$= [-\frac{8}{e}, -12, 1]$

Tangent Plane: $\boxed{-\frac{8}{e}x - 12y + z = d}$

Sub $(x, y, z) = (e, 2, 8)$: $-8 - 24 + 8 = d$

$d = -24$

$$\boxed{-\frac{8}{e}x - 12y + z = -24}$$

② 1) Interior Critical Points

$$\left.\begin{array}{l} T_x = 4x \\ T_y = 2y-1 \end{array}\right\} = 0 \text{ or undefined}$$

$$\Rightarrow (x,y) = (0, \tfrac{1}{2})$$

$$T(0, \tfrac{1}{2}) = -\tfrac{1}{4}$$

2) Critical Points on the Boundary

$$x^2+y^2=1 \Rightarrow x^2 = 1-y^2$$

Sub $x^2=1-y^2$ into T : $T = 2(1-y^2)+y^2-y$

$$= 2-y^2-y$$

$$\frac{dT}{dy} = -2y-1 = 0$$

$$\Rightarrow y = \frac{-1}{2}$$

$$\Rightarrow x^2 = 1-\frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow x = \frac{\pm\sqrt{3}}{2}$$

$$T(\pm\tfrac{\sqrt{3}}{2}, \tfrac{-1}{2}) = 2-y^2-y\Big|_{y=\frac{-1}{2}} = \frac{9}{4}$$

$$\boxed{\begin{array}{l}\text{The hottest points are } (\pm\frac{\sqrt{3}}{2}, \frac{-1}{2}) \text{ with } T=\frac{9}{4} \\ \text{the coldest point is } (0, \frac{1}{2}) \text{ with } T=-\frac{1}{4}\end{array}}$$

(3) $dm = \frac{\partial m}{\partial E} dE + \frac{\partial m}{\partial v} dv$

$dm = \frac{2}{v^2} dE - \frac{4E}{v^3} dv$

$\frac{dm}{m} = \frac{2}{v^2} \cdot \frac{v^2}{2E} dE - \frac{4E}{v^3} \cdot \frac{v^2}{2E} dv$

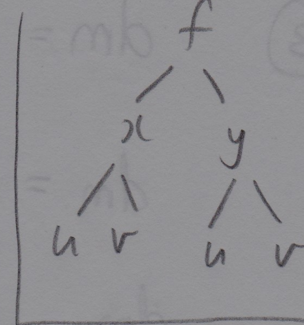
$\frac{dm}{m} = \frac{dE}{E} - 2\frac{dv}{v}$

$$\boxed{\text{For } \underline{\text{maximum}} \ \frac{dm}{m}, \text{ use } \frac{dE}{E} = 2\% \text{ and } \frac{dv}{v} = -3\%}$$

Maximum $\frac{dm}{m} = 2\% - 2(-3\%)$

$= 8\%$

(4) $$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$



$$= 4x^3(2u) - 4y^3(1)$$

$$\boxed{\begin{array}{l}\text{When } u=1,\ v=-1:\\ \quad x=2 \quad y=3\end{array}}$$

Subbing in: $$\frac{\partial f}{\partial u} = 4(2^3)(2) - 4(3^3)(1)$$
$$= -44$$

(5) z is a function of y

Take $\frac{\partial}{\partial y}$:

$$3x^2 \frac{\partial z}{\partial y} - 2x^2 y + 6z^2 \frac{\partial z}{\partial y} + \underbrace{3y \frac{\partial z}{\partial y} + 3z}_{\text{product rule}} = 0$$

$$(3x^2 + 6z^2 + 3y)\frac{\partial z}{\partial y} = 2x^2 y - 3z$$

$$\frac{\partial z}{\partial y} = \frac{2x^2 y - 3z}{3x^2 + 6z^2 + 3y}$$

(6) $\nabla T = [-8x, -2y]$

$\nabla T(2,-3) = [-16, 6]$

The direction of maximum rate of increase is $[-16, 6]$

The maximum rate of increase is

$\| \nabla T \| = \| [-16, 6] \| = \sqrt{292}$ °C/cm

⑦ $\nabla\phi = [10x - 3y + yz,\ -3x + xz,\ xy]$

$\nabla\phi(3,4,5) = [38,\ 6,\ 12]$

$\bar{u} = \frac{1}{\sqrt{3}}[1, 1, -1]$

$D_{\bar{u}}\phi = \nabla\phi \cdot \bar{u}$

$= \frac{1}{\sqrt{3}}[38, 6, 12] \cdot [1, 1, -1]$

$= \frac{32}{\sqrt{3}}$ or $\frac{32\sqrt{3}}{3}$ V/m

⑧ Surface $f=c$ has normal vector ∇f

$$\nabla f = [2x, -4y+z, 2z+y]$$

$$\nabla f(2,1,-1) = [4, -5, -1]$$

$$\bar{n} = [4, -5, -1]$$

Tangent plane: $\boxed{4x - 5y - z = d}$

Sub $(2,1,-1)$: $8 - 5 + 1 = d$

$d = 4$

$$\boxed{4x - 5y - z = 4}$$

(9) Want to minimize $\sqrt{x^2+y^2+z^2}$

It's equivalent to minimize $f=x^2+y^2+z^2$

Subject to $g=c$: $\underbrace{4xy+3-z}_{g}=0$

$$\nabla f = \lambda \nabla g$$

$$[2x, 2y, 2z] = \lambda[4y, 4x, -1]$$

$$\begin{cases} 2x = 4\lambda y \Rightarrow \lambda = \frac{x}{2y} \\ 2y = 4\lambda x \Rightarrow \lambda = \frac{y}{2x} \\ 2z = -\lambda \Rightarrow \lambda = -2z \end{cases}$$

$$\lambda = \frac{x}{2y} = \frac{y}{2x} = -2z$$

$$2x^2 = 2y^2$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow \boxed{y = \pm x}$$

$$\boxed{z = \frac{-x}{4y}}$$

Case 1. $y = x \Rightarrow z = -\frac{1}{4}$

Sub into g: $4x^2+3+\frac{1}{4}=0$ NO SOLUTION

$\rightarrow$

⑨ Cont'd

Case 2. $y = -x \Rightarrow z = \frac{1}{4}$

Sub into g : $-4x^2 + 3 - \frac{1}{4} = 0$

$$-4x^2 = -\frac{11}{4}$$

$$x^2 = \frac{11}{16}$$

$$x = \frac{\pm\sqrt{11}}{4}$$

Two solutions :

$\left(\frac{\sqrt{11}}{4}, \frac{-\sqrt{11}}{4}, \frac{1}{4}\right)$

and $\left(\frac{-\sqrt{11}}{4}, \frac{\sqrt{11}}{4}, \frac{1}{4}\right)$

(10) $z_x = 6xy - 6x = 6x(y-1) = 0$

$\Rightarrow x = 0$ or $y = 1$

$z_y = 3x^2 + 3y^2 - 6y$

$\boxed{x=0}$

$z_y = 3y^2 - 6y$
$= 3y(y-2)$
$= 0$

$\Rightarrow y = 0, 2$

$\boxed{y=1}$

$z_y = 3x^2 - 3$
$= 3(x^2-1)$
$= 3(x-1)(x+1)$
$= 0$

$\Rightarrow x = \pm 1$

4 Critical Points : $(0,0), (0,2), (\pm 1, 1)$

Recall $z_x = 6xy - 6x$ $\quad z_y = 3x^2 + 3y^2 - 6y$

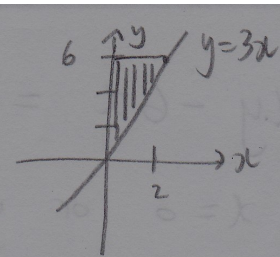
$z_{xx} = 6y - 6 \quad z_{xy} = 6x \quad z_{yy} = 6y - 6$

$$\Delta = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6y-6 & 6x \\ 6x & 6y-6 \end{vmatrix} = (6y-6)^2 - 36x^2$$

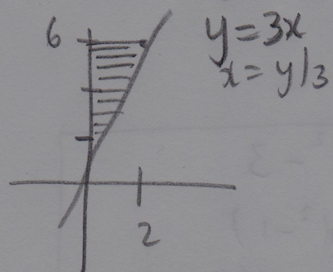
Point	Δ	z_{xx}	Type of Point
$(0,0)$	>0	<0	Local Max
$(0,2)$	>0	>0	Local Min
$(1,1)$	<0		Saddle Point
$(-1,1)$	<0		Saddle Point

(11) $3x \le y \le 6$

$0 \le x \le 2$



Horizontal Slices:



$0 \le x \le \frac{y}{3}$

$0 \le y \le 6$

$$\text{Integral} = \int_0^6 \int_0^{y/3} e^{-y^2} \, dx \, dy$$

$$= \int_0^6 x e^{-y^2} \Big|_{x=0}^{x=y/3} \, dy$$

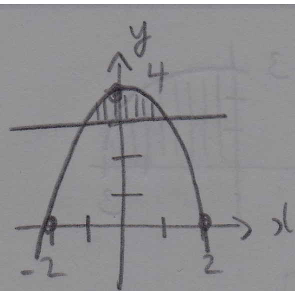
$$= \int_0^6 \frac{y}{3} e^{-y^2} \, dy$$

$$= -\frac{1}{6} e^{-y^2} \Big|_0^6$$

$$= -\frac{1}{6} e^{-36} + \frac{1}{6}$$

$$= \frac{1}{6}\left(1 - e^{-36}\right)$$

(12) a)



$\leftarrow$ Intersection

$4-x^2=3$

$1=x^2$

$x=\pm 1$

$$A=\int_{-1}^{1}\int_{3}^{4-x^2} dy\,dx$$

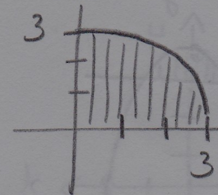
b) Over this region, $2+x<4$

$$V=\int_{-1}^{1}\int_{3}^{4-x^2}(4-(2+x))\,dy\,dx$$

$$=\int_{-1}^{1}\int_{3}^{4-x^2}(2-x)\,dy\,dx$$

(13) $0 \le y \le \sqrt{9-x^2}$

$0 \le x \le 3$



$$\boxed{\rightarrow \text{Polar} \quad 0 \le r \le 3, \quad 0 \le \theta \le \frac{\pi}{2}}$$

$dA = r\,dr\,d\theta$

$$\text{Integral} = \int_0^{\pi/2}\int_0^3 (\sin r^2)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[-\frac{1}{2}\cos r^2\right]_0^3 d\theta$$

$$= \frac{\pi}{2}\left(-\frac{1}{2}\right)(\cos 9 - \cos 0)$$

$$= -\frac{\pi}{4}(\cos 9 - 1)$$

$$= \frac{\pi}{4}(1 - \cos 9)$$