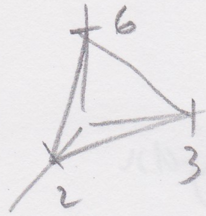
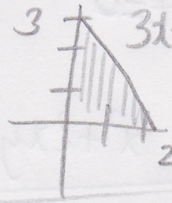
36

a)



$0 \le z \le 6 - 3x - 2y$



$3x + 2y = 6 \Rightarrow 0 \le y \le 3 - \frac{3}{2}x$

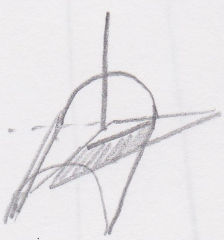
$0 \le x \le 2$

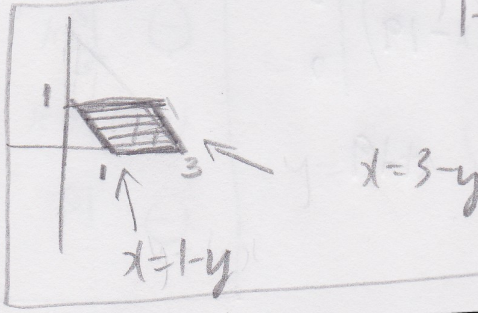
[redacted]

ANSWER: $\displaystyle\int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} f(x,y,z)\,dz\,dy\,dx$

b)

$0 \le z \le 1 - y^2$





$1 - y \le x \le 3 - y$

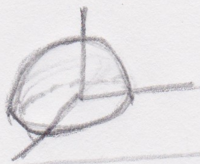
$0 \le y \le 1$

[redacted]

ANSWER: $\displaystyle\int_0^1 \int_{1-y}^{3-y} \int_0^{1-y^2} f(x,y,z)\,dz\,dx\,dy$

$\rightarrow$

(36) c)



$$0 \le z \le \sqrt{1-x^2-y^2}$$

$x^2+y^2=1$

$y=\pm\sqrt{1-x^2}$

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

$$-1 \le x \le 1$$

ANSWER: [redacted]

$$\boxed{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx}$$

d)

$$x^2+y^2 \le z \le \sqrt{6-x^2-y^2}$$

Intersection

$x^2+y^2=\sqrt{6-x^2-y^2}$

$(x^2+y^2)^2 = 6-x^2-y^2$

$r^4 = 6-r^2$

$r^4+r^2-6=0$

$(r^2+3)(r^2-2)=0$

$r^2=2 \qquad \cancel{r^2=-3}$

$x^2+y^2=2 \implies$

$$-\sqrt{2-x^2} \le y \le \sqrt{2-x^2}$$

$$-\sqrt{2} \le x \le \sqrt{2}$$

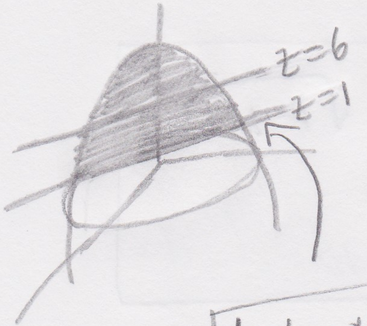
[redacted]

ANSWER:

$$\boxed{\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx}$$

(37) SA_1 = Surface Area above $z=1$

$$= \iint_R \sqrt{1+z_x^2+z_y^2}\,dy\,dx$$



$$= \iint_R \sqrt{1+4x^2+4y^2}\,dy\,dx$$

$\rightarrow$ Polar

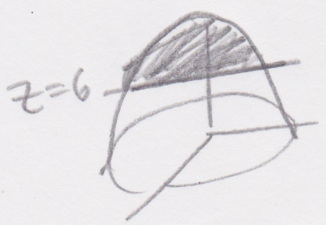
$$\boxed{\begin{aligned}1&=10-x^2-y^2\\ x^2+y^2&=9\end{aligned}}$$

$$= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$= 2\pi\left[\frac{1}{8}\frac{2}{3}(1+4r^2)^{3/2}\right]_0^3$$

$$= \frac{\pi}{6}\left[37^{3/2}-1\right]$$

SA_2 = Surface Area above $z=6$



$$\boxed{\begin{aligned}6&=10-x^2-y^2\\ x^2+y^2&=4\end{aligned}}$$

$$SA_2 = \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$= 2\pi\left[\frac{1}{8}\frac{2}{3}(1+4r^2)^{3/2}\right]_0^2$$

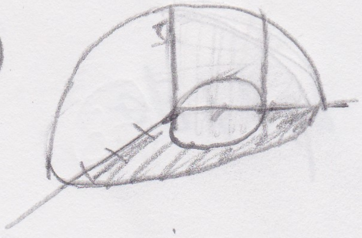
$$= \frac{\pi}{6}\left[17^{3/2}-1\right]$$

Surface Area between $z=1$ and $z=6$

$$= SA_1 - SA_2$$

$$= \frac{\pi}{6}\left[37^{3/2}-17^{3/2}\right]$$

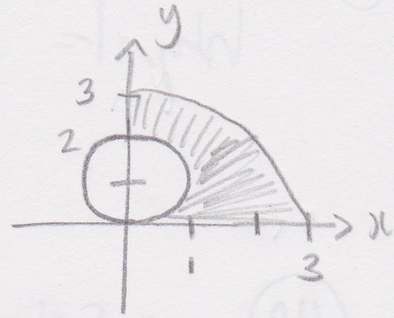
(38)



$x^2+y^2=2y$

$r^2=2r\sin\theta$

$r=2\sin\theta$



$$\boxed{\begin{aligned} &2\sin\theta \le r \le 3 \\ &0 \le \theta \le \pi/2 \\ &0 \le z \le \sqrt{9-r^2} \\ &dV = r\,dz\,dr\,d\theta \\ &x^2+y^2=r^2 \end{aligned}}$$

$$\text{Integral} = \int_0^{\pi/2}\int_{2\sin\theta}^{3}\int_0^{\sqrt{9-r^2}} r^3\,dz\,dr\,d\theta$$

(39)

$\boxed{1 \le \rho \le 3}$

$\boxed{0 \le \theta \le 2\pi}$

$z = 2\sqrt{x^2+y^2}$

$z = 2r$

$\rho\cos\phi = 2\rho\sin\phi$

$\frac{1}{2} = \tan\phi$

$\boxed{0 \le \phi \le \arctan\frac{1}{2}}$

$x^2+y^2 = r^2 = \rho^2\sin^2\phi$

$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

→

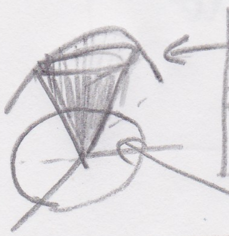
(39) Cont'd

$$\text{Integral} = \int_0^{2\pi}\int_0^{\arctan\frac{1}{2}}\int_1^3 \rho^4 \sin^3\phi \quad d\rho\, d\phi\, d\theta$$

(40) → Spherical

Integrand = $\sqrt{x^2+y^2+z^2} = \rho$

$$dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$



$z = \sqrt{2-x^2-y^2}$
$z^2 = 2-x^2-y^2$
$\rho = \sqrt{2}$

$z = r$
$\phi = \pi/4$

$0 \le \rho \le \sqrt{2}$
$0 \le \phi \le \frac{\pi}{4}$
$0 \le \theta \le 2\pi$

$$\text{Integral} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^3 \sin\phi \, d\rho\, d\phi\, d\theta$$

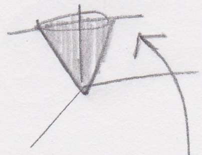
$$= \int_0^{2\pi}\int_0^{\pi/4} \sin\phi \, d\phi\, d\theta$$

$$= 2\pi \left[-\cos\phi\right]_0^{\pi/4}$$

$$= 2\pi \left[1 - \frac{1}{\sqrt{2}}\right]$$

$$= \pi\left[2 - \sqrt{2}\right]$$

(41)



$$\boxed{\begin{aligned} z &= 3 \\ \rho\cos\phi &= 3 \\ \rho &= 3\sec\phi \end{aligned}}$$

$$\boxed{\begin{aligned} z &= \sqrt{x^2+y^2} \\ z &= r \\ \rho\cos\phi &= \rho\sin\phi \\ 1 &= \tan\phi \\ \phi &= \pi/4 \end{aligned}}$$

$$\boxed{\delta = \rho}$$

$$m = \iiint_Q \delta \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{3\sec\phi} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\boxed{\frac{81}{4}(\cos\phi)^{-4}}$$

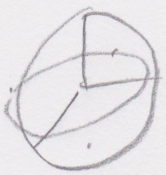
$$= 2\pi \int_0^{\pi/4} \frac{81\sec^4\phi}{4} \sin\phi \, d\phi$$

$$= 2\pi \left[\frac{27}{4} (\cos\phi)^{-3} \right]_0^{\pi/4}$$

$$= 2\pi \left[\frac{27}{4} \sqrt{2}^3 - \frac{27}{4} \right]$$

$$= \frac{27}{2}\pi \left[2\sqrt{2} - 1 \right]$$

(42)



$$\boxed{\begin{aligned} x^2+y^2 &= r^2 \\ &= \rho^2\sin^2\phi \end{aligned}}$$

$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

$$\iiint_B (x^2+y^2) \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{1} \rho^2 \sin^2\phi \; \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{1} \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta$$

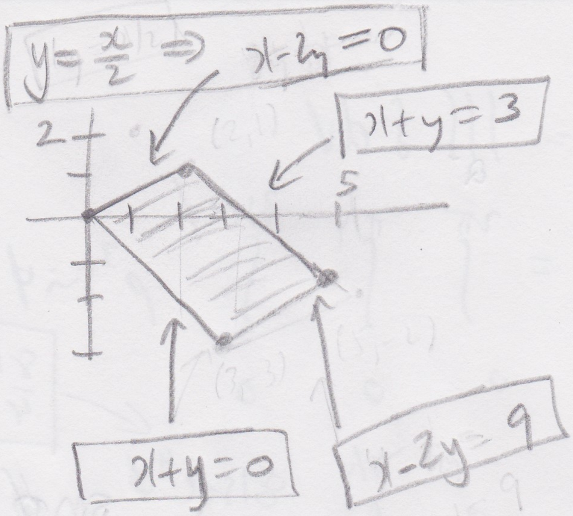
$$= \frac{1}{5} \int_0^{2\pi} \int_0^{\pi} \sin^3\phi \, d\phi \, d\theta \quad \longleftarrow \boxed{\sin^3\phi = (1-\cos^2\phi)\sin\phi}$$

$$= \frac{2\pi}{5} \left[-\cos\phi + \frac{\cos^3\phi}{3} \right]_0^{\pi}$$

$$= \frac{2\pi}{5} \left[1 - \frac{1}{3} + \frac{2}{3} \right]$$

$$= \frac{8\pi}{15}$$

43



$$\boxed{\begin{array}{ll} u = x+y & 0 \le u \le 3 \\ v = x-2y & 0 \le v \le 9 \end{array}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = -\frac{1}{3}$$

$$\iint_R (x+y)\,dA = \iint_R u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \frac{1}{3}\int_0^9\int_0^3 u\,du\,dv$$

$$= \frac{1}{3}\int_0^9 \frac{9}{2}\,dv$$

$$= \frac{1}{3}\left(\frac{81}{2}\right) = \frac{27}{2}$$

(44) $u = xy \quad v = \dfrac{y}{x}$

$$\boxed{\begin{array}{l} 1 \le u \le 2 \\ 1 \le v \le 3 \end{array}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{2y}{x} = 2v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2v} \text{ since } v>0 \qquad \boxed{y^2 = uv}$$

$$\iint_R y^2\,dA = \int_1^3\int_1^2 uv\frac{1}{2v}\,du\,dv$$

$$= \int_1^3\int_1^2 \frac{u}{2}\,du\,dv$$

$$= \int_1^3 \frac{u^2}{4}\Big|_1^2\,dv$$

$$= \frac{3}{4}(2)$$

$$= \frac{3}{2}$$

(45) $\nabla \cdot \vec{F} = 2xyz + 3xz^3 - 2z$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & 3xyz^3 & x^2-z^2 \end{vmatrix}$$

$$= \vec{i}\left(-9xyz^2\right) - \vec{j}\left(2x - x^2y\right) + \vec{k}\left(3yz^3 - x^2z\right)$$

$$= -9xyz^2\vec{i} + (x^2y - 2x)\vec{j} + (3yz^3 - x^2z)\vec{k}$$

(46) $$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\cos y + yz & xz - e^x\sin y & xy \end{vmatrix}$$

$$= \vec{i}\,(x - x) - \vec{j}\,(y - y) + \vec{k}\,(z - z)$$

$$= \vec{0}$$

$\Rightarrow \vec{F}$ is conservative

Cont'd $\rightarrow$

(46) Cont'd

$$f = \int (e^x \cos y + yz)\,dx$$

$$= e^x \cos y + xyz + g(y,z)$$

and

$$f = \int (xz - e^x \sin y)\,dy$$

$$= xyz + e^x \cos y + h(x,z)$$

and

$$f = \int xy\,dz$$

$$= xyz + k(x,y)$$

$$\Rightarrow f = e^x \cos y + xyz + C$$

(47)

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2}\, dt$$

$$= \sqrt{9+16}\, dt$$

$$= 5\, dt$$

$$\int_C z\, ds = \int_0^{\pi} 4t \cdot 5\, dt$$

$$= 10t^2 \Big|_0^{\pi}$$

$$= 10\pi^2$$

(48)

$$\bar{x} = \bar{p} + t\bar{d} \qquad \bar{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\boxed{\begin{aligned} x &= 1+t \\ y &= -1+2t \\ z &= 2+3t \\ 0 &\le t \le 1 \end{aligned}}$$

$$ds = \sqrt{1+4+9}\, dt$$

$$= \sqrt{14}\, dt$$

$$\int_C (x+3y+2z)\, ds = \int_0^1 \left[(1+t) + 3(-1+2t) + 2(2+3t)\right]\sqrt{14}\, dt$$

$$= \sqrt{14} \int_0^1 (2+13t)\, dt$$

$$= \sqrt{14} \left[2t + \frac{13t^2}{2} \right]_0^1 = \frac{17\sqrt{14}}{2}$$

(49)

$$\boxed{\begin{array}{lll} x = t^2 & y = t^3 & 0 \le t \le 1 \\ dx = 2t\,dt & dy = 3t^2\,dt & \end{array}}$$

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_C (P\,dx + Q\,dy) \\
&= \int_C (xy\,dx + x^2\,dy) \\
&= \int_0^1 t^5(2t\,dt) + t^4(3t^2\,dt) \\
&= \int_0^1 5t^6\,dt \\
&= \frac{5t^7}{7}\Big|_0^1 \\
&= \frac{5}{7}
\end{aligned}$$

(50) $\text{curl}\,\vec{F} = \vec{0} \Rightarrow \vec{F}$ is conservative

$f = x^4 + y^2 z^3$ is a potential function

Fundamental Theorem of Line Integrals $\Rightarrow$

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$$

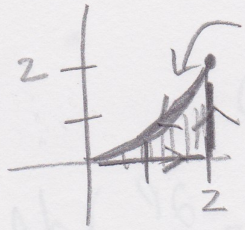
endpoints of C

$$\boxed{\begin{array}{l} A = \vec{r}(0) = (1,-1,1) \\ B = \vec{r}(2) = (3,3,3) \end{array}}$$

$$= f(3,3,3) - f(1,-1,1)$$

$$= 324 - 2 = 322$$

(51)



$y = x^3/4$

Green's Theorem $\Rightarrow$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^2 \int_0^{\frac{x^3}{4}} (2x - 2y)\,dy\,dx$$

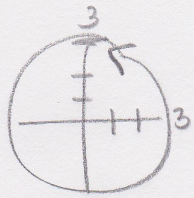
$$= \int_0^2 2xy - y^2 \Big|_0^{y=\frac{x^3}{4}} dx$$

$$= \int_0^2 \Big[\underbrace{2x\left(\frac{x^3}{4}\right)}_{\frac{x^4}{2}} - \frac{x^6}{16}\Big]\,dx$$

$$= \frac{x^5}{10} - \frac{x^7}{112} \Big|_0^2$$

$$= \frac{72}{35}$$

(52)



Green's Theorem $\Rightarrow$

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R 3\, dA$$

$$= 3 \iint_R dA$$

$$= 3 \cdot \text{area of circle}$$

$$= 3 \cdot 9\pi$$

$$= 27\pi$$

(53)

$$SA = \iint_S \|\vec{r}_u \times \vec{r}_v\|\, du\, dv$$

$$\vec{r}_u = [a\cos u \cos v,\ a\cos u \sin v,\ -a\sin u]$$

$$\vec{r}_v = [-a\sin u \sin v,\ a\sin u \cos v,\ 0]$$

$$\vec{r}_u \times \vec{r}_v = [a^2\sin^2 u \cos v,\ a^2 \sin^2 u \sin v,\ a^2 \cos u \sin u]$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{a^4 \sin^4 u \cos^2 v + a^4 \sin^4 u \sin^2 v + a^4 \cos^2 u \sin^2 u}$$

$$= \sqrt{a^4 \sin^4 u + a^4 \sin^2 u \cos^2 u}$$

$$= \sqrt{a^4 \sin^2 u (\sin^2 u + \cos^2 u)}$$

$$= \sqrt{a^4 \sin^2 u}$$

$$= a^2 \sin u \qquad (\geq 0 \text{ on } 0 \leq u \leq \pi)$$

$\longrightarrow$

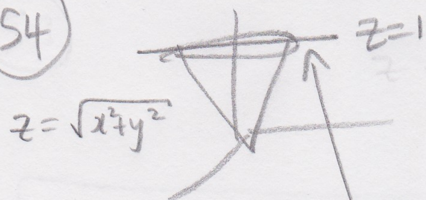
(53) Cont'd

$$SA = \int_0^{2\pi}\int_0^{\pi} a^2 \sin u \, du\, dv$$

$$= 2\pi a^2 \left[-\cos u\right]_0^{\pi}$$

$$= 4\pi a^2$$

(54)



$z = z$

$\sqrt{x^2+y^2} = 1$

$r = 1$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

Position on S:

$\vec{r}(x,y) = [x, y, \sqrt{x^2+y^2}]$

$\vec{r}(r,\theta) = [r\cos\theta, r\sin\theta, r]$

$\vec{r}_r = [\cos\theta, \sin\theta, 1]$

$\vec{r}_\theta = [-r\sin\theta, r\cos\theta, 0]$

$\vec{r}_r \times \vec{r}_\theta = [-r\cos\theta, -r\sin\theta, r]$

$\|\vec{r}_r \times \vec{r}_\theta\| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}$

$= \sqrt{r^2+r^2}$

$= \sqrt{2}r$ $\quad \boxed{dS = \sqrt{2}r\,dr\,d\theta}$

$\boxed{\text{Integrand} = z = r}$

$$\iint_S z\,dS = \int_0^{2\pi}\int_0^{1} \sqrt{2} r^2 \,dr\,d\theta$$

$\rightarrow$

(54) Cont'd

$$= 2\pi \cdot \left[\frac{\sqrt{2}r^3}{3}\right]_0^1$$

$$= \frac{2\sqrt{2}\pi}{3}$$

(55) Use the Divergence Theorem:

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_T \operatorname{div}\vec{F}\,dV$$

$z=8$

$z=z$

$2(x^2+y^2)=8$

$x^2+y^2=4$

$r^2=4$

$r=2$

$\Rightarrow\ 2r^2 \le z \le 8$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

(cylindrical)

$$\boxed{\begin{aligned}\operatorname{div}\vec{F} &= 3x^2+3y^2\\ &= 3r^2\\ dV &= r\,dz\,dr\,d\theta\end{aligned}}$$

$$\text{Flux } \Phi = \int_0^{2\pi}\int_0^{2}\int_{2r^2}^{8} 3r^2 \cdot r\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^2 3r^3(8-2r^2)\,dr$$

$$= 2\pi \int_0^2 (24r^3 - 6r^5)\,dr$$

$$= 2\pi\left[6r^4 - r^6\right]_0^2$$

$$= 2\pi\,[32]$$

$$= 64\pi$$