(25) Minimize $\sqrt{x^2+y^2}$

Equivalently: minimize $f = x^2+y^2$

Subject to $g=c$: $x^2y=16$

$$\nabla f = \lambda \nabla g$$

$$[2x, 2y] = \lambda[2xy, x^2]$$

$$\begin{cases} 2x = \lambda 2xy & \Rightarrow \lambda = \frac{1}{y} \quad (x \neq 0 \text{ because } x^2y=16) \\ 2y = \lambda x^2 & \Rightarrow \lambda = \frac{2y}{x^2} \end{cases}$$

$$\lambda = \frac{1}{y} = \frac{2y}{x^2}$$

$$x^2 = 2y^2$$

$$x = \pm\sqrt{2}y$$

$\rightarrow x^2y = 16$: $2y^2 \cdot y = 16$

$$y^3 = 8$$

$$y = 2$$

$$\Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$$

Closest points are $(\pm 2\sqrt{2}, 2)$

(26) Minimize and maximize $f = 3x + 2y + z$

Subject to $g = c$: $9x^2 + 4y^2 - z = 0$

$$\nabla f = \lambda \nabla g$$

$$[3, 2, 1] = \lambda [18x, 8y, -1]$$

Third Component: $1 = -\lambda \Rightarrow \lambda = -1$

$$3 = \lambda 18x \Rightarrow x = \frac{-1}{6}$$

$$2 = \lambda 8y \Rightarrow y = \frac{-1}{4}$$

$$9x^2 + 4y^2 - z = 0 \Rightarrow \frac{1}{4} + \frac{1}{4} - z = 0$$
$$\Rightarrow z = \frac{1}{2}$$

$(x, y, z) = \left(-\frac{1}{6}, \frac{-1}{4}, \frac{1}{2}\right)$ yields a minimum value of $f\left(\frac{-1}{6}, \frac{-1}{4}, \frac{1}{2}\right) = -1/2$

As $x, y \to \infty$, $z \to \infty$ and $f \to \infty$

f does not have a maximum value.

(27) Minimize and maximize $\sqrt{x^2+y^2}$

Equivalently, min/max $f = x^2+y^2$

subject to $g=c$: $17x^2+12xy+8y^2=100$

$$\nabla f = \lambda \nabla g$$

$$[2x, 2y] = \lambda [34x+12y, 12x+16y]$$

$$\begin{cases} 2x = \lambda(34x+12y) \Rightarrow \lambda = \dfrac{x}{17x+6y} \\ 2y = \lambda(12x+16y) \Rightarrow \lambda = \dfrac{y}{6x+8y} \end{cases}$$

$$\Rightarrow \frac{x}{17x+6y} = \frac{y}{6x+8y}$$

$$6x^2+8xy = 17xy+6y^2$$

$$\boxed{6x^2-9xy-6y^2=0}$$

$$\begin{cases} 17x^2+12xy+8y^2=100 & (1) \\ 6x^2-9xy-6y^2=0 & (2) \end{cases}$$

$$\begin{array}{rl} 3\times(1) & 51x^2+36xy+24y^2=300 \\ 4\times(2) + & 24x^2-36xy-24y^2=0 \\ \hline & 75x^2 = 300 \end{array} \Rightarrow x = \pm 2$$

$\rightarrow$

(27) Cont'd

$x=2 \to$ ② : $\quad 24 - 18y - 6y^2 = 0$

$$y^2 + 3y - 4 = 0$$
$$(y+4)(y-1) = 0$$
$$y = -4, 1$$

$x=-2 \to$ ② : $\quad 24 + 18y - 6y^2 = 0$

$$y^2 - 3y - 4 = 0$$
$$(y-4)(y+1) = 0$$
$$y = 4, -1$$

Closest points $(2,1)$, $(-2,-1)$ have $d = \sqrt{5}$

Furthest points $(2,-4)$, $(-2,4)$ have $d = 2\sqrt{5}$

(28) Minimize $f = \sqrt{x^2+y^2+z^2}$

Equivalently: minimize $f = x^2+y^2+z^2$

subject to $\underbrace{xy+5-z}_{g} = 0$

$$\nabla f = \lambda \nabla g$$

$$[2x, 2y, 2z] = \lambda [y, x, -1]$$

$$\left.\begin{aligned} 2x &= \lambda y \\ 2y &= \lambda x \\ 2z &= -\lambda \end{aligned}\right\}$$

$$\lambda = \frac{2x}{y} = \frac{2y}{x} = -2z$$

$$2x^2 = 2y^2$$

$$x^2 = y^2$$

$$y = \pm x$$

$$\frac{x}{y} = -z$$

Case 1: $y = x \Rightarrow z = -1$

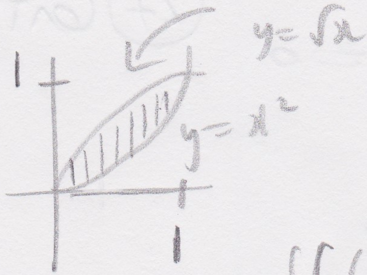
Plug into g: $x^2+6=0$ No solution

Case 2: $y = -x \Rightarrow z = 1$

Plug into g: $-x^2+4=0 \Rightarrow x = \pm 2$

Two solutions: $(x,y,z) = (2,-2,1)$ or $(-2,2,1)$

(29)



$$\iint_R (x+y)\,dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_{y=x^2}^{y=\sqrt{x}} dx$$

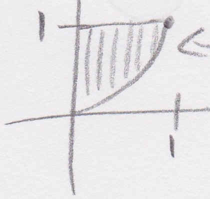
$$= \int_0^1 \left(x^{3/2} + \frac{x}{2} - x^3 - \frac{x^4}{2} \right) dx$$

$$= \left[\frac{2}{5}x^{5/2} + \frac{x^2}{4} - \frac{x^4}{4} - \frac{x^5}{10} \right]_0^1$$

$$= \frac{2}{5} - \frac{1}{10}$$

$$= 3/10$$

(30)



$$m = \iint_R \delta \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^1 xy \, dy \, dx$$

$$= \int_0^1 \frac{xy^2}{2} \Big|_{y=x^2}^{y=1} dx$$

$$= \int_0^1 \left(\frac{x}{2} - \frac{x^5}{2} \right) dx$$

$$= \left[\frac{x^2}{4} - \frac{x^6}{12} \right]_0^1$$

$$= \frac{1}{6}$$

$$\bar{x} = \frac{1}{m} \iint_R x \delta \, dy \, dx$$

$$= 6 \int_0^1 \int_{x^2}^1 x^2 y \, dy \, dx$$

$$= 6 \int_0^1 \frac{x^2 y^2}{2} \Big|_{y=x^2}^{y=1} dx$$

$$= 6 \int_0^1 \left(\frac{x^2}{2} - \frac{x^6}{2} \right) dx$$

$$= 6 \left[\frac{x^3}{6} - \frac{x^7}{14} \right]_0^1$$

$$= 6 \left[\frac{1}{6} - \frac{1}{14} \right]$$

$$= 4/7$$

$\rightarrow$

30 Cont'd

$$\bar{y} = \frac{1}{m} \iint_R y\,\delta \, dy\, dx$$

$$= 6 \int_0^1 \int_{x^2}^1 x y^2 \, dy\, dx$$

$$= 6 \int_0^1 \frac{x y^3}{3} \Bigg|_{y=x^2}^{y=1} dx$$

$$= 6 \int_0^1 \left(\frac{x}{3} - \frac{x^7}{3} \right) dx$$

$$= 6 \left[\frac{x^2}{6} - \frac{x^8}{24} \right]_0^1$$

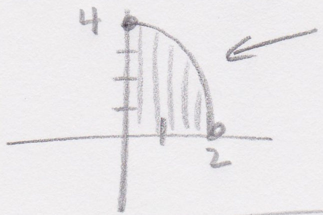
$$= 6 \left[\frac{1}{6} - \frac{1}{24} \right]$$

$$= 3/4$$

$$(\bar{x}, \bar{y}) = \left(\frac{4}{7}, \frac{3}{4} \right)$$

(31) Change order of integration

a) $\boxed{\begin{array}{l} 0 \le y \le 4-x^2 \\ 0 \le x \le 2 \end{array}}$



$y = 4-x^2$

$x^2 = 4-y$

$x = \pm\sqrt{4-y}$

$x = \sqrt{4-y}$

$\boxed{\begin{array}{l} 0 \le x \le \sqrt{4-y} \\ 0 \le y \le 4 \end{array}}$

$$I = \int_0^4 \int_0^{\sqrt{4-y}} \frac{x e^{2y}}{4-y}\, dx\, dy$$

$$= \int_0^4 \frac{x^2 e^{2y}}{2(4-y)} \Bigg|_{x=0}^{x=\sqrt{4-y}} dy$$

$$= \int_0^4 \frac{\cancel{(4-y)} e^{2y}}{2\cancel{(4-y)}}\, dy$$

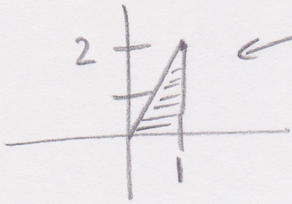
$$= \frac{e^{2y}}{4} \Bigg|_0^4$$

$$= \frac{1}{4}(e^8 - 1)$$

→

(31) b)

$$\boxed{\begin{aligned} \frac{y}{2} &\le x \le 1 \\ 0 &\le y \le 2 \end{aligned}}$$



$x = y/2$

$y = 2x$

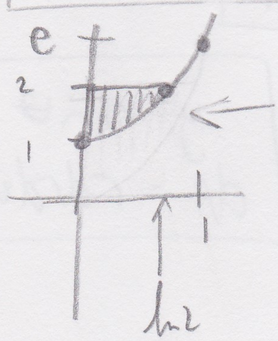
$$\boxed{\begin{aligned} 0 &\le y \le 2x \\ 0 &\le x \le 1 \end{aligned}}$$

$$\begin{aligned}
I &= \int_0^1 \int_0^{2x} \sin x^2 \, dy \, dx \\
&= \int_0^1 y \sin x^2 \Big|_{y=0}^{y=2x} dx \\
&= \int_0^1 2x \sin x^2 \, dx \\
&= -\cos x^2 \Big|_0^1 \\
&= -\cos 1 + 1 \\
&= 1 - \cos 1
\end{aligned}$$

$\rightarrow$

31 c)

$$\boxed{\begin{array}{l} e^x \le y \le 2 \\ 0 \le x \le \ln 2 \end{array}}$$



$y = e^x$

$x = \ln y$

$$\boxed{\begin{array}{l} 0 \le x \le \ln y \\ 1 \le y \le 2 \end{array}}$$

$$I = \int_1^2 \int_0^{\ln y} \frac{1}{\ln y} \, dx \, dy$$

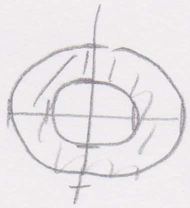
$$= \int_1^2 \frac{x}{\ln y} \Big|_{x=0}^{x=\ln y} dy$$

$$= \int_1^2 1 \, dy$$

$$= y \Big|_1^2$$

$$= 1$$

(32)



$$\boxed{\begin{array}{l} 1 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{array}}$$

$$\iint_R y^2 \, dA$$

$$\boxed{\begin{array}{l} y = r\sin\theta \\ dA = r\,dr\,d\theta \end{array}}$$

$$= \int_0^{2\pi}\int_1^2 r^3 \sin^2\theta \, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{r^4 \sin^2\theta}{4}\Bigg|_{r=1}^{r=2} d\theta$$

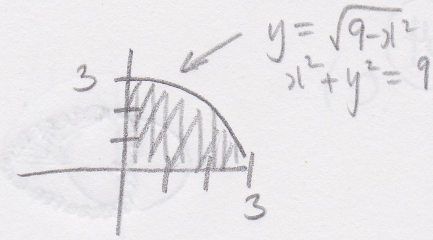
$$= \int_0^{2\pi} \frac{15}{4}\sin^2\theta \, d\theta$$

$$= \int_0^{2\pi} \frac{15}{8}(1-\cos 2\theta)\, d\theta$$

$$= \left[\frac{15}{8}\theta - \frac{15\sin 2\theta}{16}\right]_0^{2\pi}$$

$$= \frac{15\pi}{4}$$

(33) $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$



$$\boxed{\begin{aligned} &0 \le r \le 3 \\ &0 \le \theta \le \tfrac{\pi}{2} \\ &x^2+y^2=r^2 \\ &dA = r\,dr\,d\theta \end{aligned}}$$

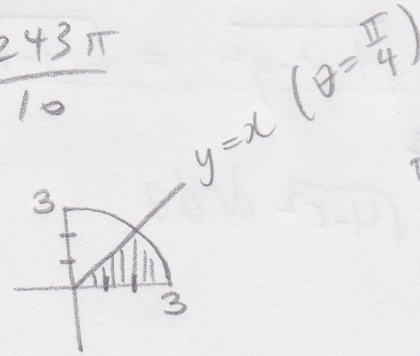
$$= \int_0^{\pi/2}\int_0^3 r^4\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{r^5}{5}\Big|_{r=0}^{r=3}\, d\theta$$

$$= \left(\frac{3^5}{5}\right)\left(\frac{\pi}{2}\right)$$

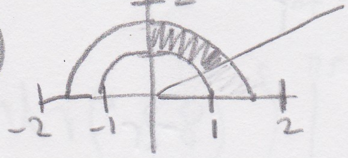
$$= \frac{243\pi}{10}$$

(34) a)



$$\int_0^{\pi/4}\int_0^3 \sqrt{1+r^2}\, r\, dr\, d\theta$$

b)



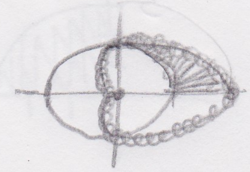
$$\boxed{\begin{aligned} &x=\sqrt{3}y \\ &y = x/\sqrt{3} \\ &r\sin\theta = r\cos\theta/\sqrt{3} \\ &\cancel{\tan\theta} = 1/\sqrt{3} \\ &\theta = \pi/6 \end{aligned}} \Rightarrow \boxed{\begin{aligned} &1 \le r \le \sqrt{2} \\ &\tfrac{\pi}{6} \le \theta \le \tfrac{\pi}{2} \end{aligned}}$$

$$\int_{\pi/6}^{\pi/2}\int_1^{\sqrt{2}} r^2\cos\theta\, dr\, d\theta$$

$\rightarrow$

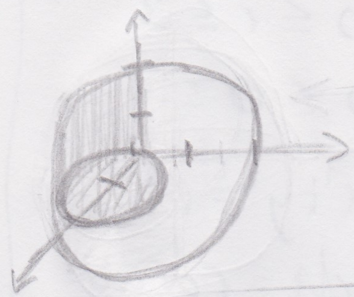
34

c)



$$\int_0^{\pi/2}\int_1^{1+\cos\theta} r^3\cos^2\theta\,dr\,d\theta$$

d)

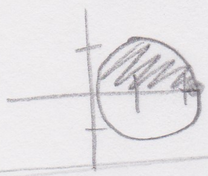


Want first octant volume

$x^2+y^2=2x$

$r^2=2r\cos\theta$

$r=2\cos\theta$



$0\le r\le 2\cos\theta$

$0\le\theta\le\frac{\pi}{2}$

$dA = r\,dr\,d\theta$

$x^2+y^2+z^2=4 \Rightarrow z=\sqrt{4-x^2-y^2}=\sqrt{4-r^2}$

$$V=\iint_R z\,dA = \int_0^{\pi/2}\int_0^{2\cos\theta} r\sqrt{4-r^2}\,dr\,d\theta$$

e)

$8-r^2=r^2$

$r^2=4$

$\Rightarrow 0\le r\le 2$

$0\le\theta\le 2\pi$

$$V=\iint_R (z_{top}-z_{bottom})\,dA$$

$$=\int_0^{2\pi}\int_0^{2}(8-2r^2)\,r\,dr\,d\theta$$

(35)

$$\int_0^2\int_0^x e^x y z + e^x z^2 \Big|_{z=0}^{z=x+y} \, dy\, dx$$

$$= \int_0^2\int_0^x \underbrace{e^x\left[y(x+y) + x^2+2xy+y^2\right]}_{e^x(x^2+3xy+2y^2)} \, dy\, dx$$

$$= \int_0^2 e^x\left(x^2y + \frac{3xy^2}{2} + \frac{2}{3}y^3\right)\Big|_{y=0}^{y=x} \, dx$$

$$= \int_0^2 e^x\left(\frac{19}{6}x^3\right) dx$$

	D	I
+	$\frac{19}{6}x^3$	e^x
−	$\frac{19}{2}x^2$	
+	$19x$	
−	19	

$$= e^x\left(\frac{19}{6}x^3 - \frac{19}{2}x^2 + 19x - 19\right)\Big|_0^2$$

$$= e^2\left(\frac{152}{6} - 38 + 38 - 19\right) + 19$$

$$= \frac{19}{3}e^2 + 19$$