Practice Solutions

① a) $x^2+y^2+(z-3)^2=9$

$\boxed{\text{Sphere of radius 3 centred at } (0,0,3)}$

b) $x^2-4x+2(y^2+2y)+z^2-2z=-3$

$x^2-4x+4+2(y^2+2y+1)+z^2-2z+1=4+2+1-3$

$(x-2)^2+2(y+1)^2+(z-1)^2=4$

$$\frac{(x-2)^2}{2^2}+\frac{(y+1)^2}{\sqrt{2}^2}+\frac{(z-1)^2}{2^2}=1$$

$\boxed{\text{Ellipsoid centred at } (2,-1,1)}$

② a) $$\lim_{(x,y)\to(1,2)}\frac{xy}{x^2+y^2}=\frac{2}{5}$$

b) Let $y=mx$

$$\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{x^2+y^2}$$

$$=\lim_{x\to0}\frac{x^2-m^2x^2}{x^2+m^2x^2}$$

$$=\lim_{x\to0}\frac{x^2(1-m^2)}{x^2(1+m^2)}$$

$$=\frac{1-m^2}{1+m^2}$$ (depends on m)

Limit d.n.e.

③ Need $x \geq 0$ and $y - 2x > 0$ and $4 - x^2 - y^2 > 0$

Domain $= \{ (x,y) \mid x \geq 0,\ y > 2x,\ x^2 + y^2 < 4 \}$

④ a) $f_y(x,y,z) = -2x^2 e^{-2y} z^3 + x^2 \cos(x^2 y)$

$f_y(3,0,2) = -18(8) + 9$

$= -135$

b) $f_x(x,y,z) = 2x e^{-2y} z^3 + 2xy \cos(x^2 y)$

$f_{xz}(x,y,z) = 6x e^{-2y} z^2$

$f_{xz}(2,-1,1) = 12e^2$

⑤ Show that $z_{xx} + z_{yy} = 0$

$z_x = y + e^{-y} \cos x$

$z_{xx} = -e^{-y} \sin x$

$z_y = x - e^{-y} \sin x$

$z_{yy} = e^{-y} \sin x$

$z_{xx} + z_{yy} = 0$ ✓

(6) $a = \frac{v^2}{r}$

$$da = \frac{\partial a}{\partial v} dv + \frac{\partial a}{\partial r} dr$$

$$da = \frac{2v}{r} dv - \frac{v^2}{r^2} dr$$

Sub $v = 10.0$ $r = 4.0$
for maximum da, use
$dv = 0.2$ and $dr = -0.1$

$$da = \frac{20.0}{4.0}(0.2) - \frac{(10.0)^2}{(4.0)^2}(-0.1)$$

$$\approx 1.6 \ m/s^2$$

$\Delta a \simeq da \simeq 1.6 \ m/s^2$ is the maximum absolute error

(7) $f(x+dx, y+dy) \approx f(x,y) + df$

where $df = f_x \, dx + f_y \, dy$

$\Rightarrow f(x+dx, y+dy) \approx f(x,y) + f_x(x,y)\,dx + f_y(x,y)\,dy$

Use $x = 3$, $y = 2$
$dx = 0.1$ $dy = -0.2$

$$f(3.1, 1.8) \simeq f(3,2) + f_x(3,2)(0.1) + f_y(3,2)(-0.2)$$

$\rightarrow$

7 Cont'd

From gradient, $f_x(3,2)=4$

$f_y(3,2)=-1$

$$\Rightarrow f(3.1,1.8) \approx 5 + 4(0.1) - (-0.2)$$
$$\approx 5.6$$

8

$$A = 2hw + \frac{16}{w} + \frac{8}{h}$$

$$dA = A_h\,dh + A_w\,dw$$
$$dA = \left(2w - \frac{8}{h^2}\right)dh + \left(2h - \frac{16}{w^2}\right)dw$$

Sub $w=1.0$ $h=4.0$

Note $2w - \frac{8}{h^2} > 0$ and $2h - \frac{16}{w^2} < 0$

for maximum error, use

$dh = 0.1$ and $dw = -0.1$

$$dA = 1.5(0.1) + (-8.0)(-0.1)$$
$$= 0.95$$

$$A\Big|_{\substack{w=1.0\\h=4.0}} = 2(4.0)(1.0) + \frac{16}{1.0} + \frac{8}{4.0}$$
$$= 26$$

Maximum relative error $\frac{dA}{A} \approx 3.65\%$

(9) $\frac{\partial z}{\partial x}$: Take $\frac{\partial}{\partial x}$ of both sides (z is a function of x)

$$2xy + z + x\frac{\partial z}{\partial x} + 3z^2\frac{\partial z}{\partial x} = 0$$

$$(x+3z^2)\frac{\partial z}{\partial x} = -2xy - z$$

$$\frac{\partial z}{\partial x} = \frac{-(2xy+z)}{x+3z^2}$$

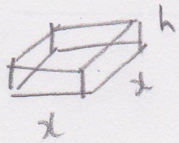
$\frac{\partial z}{\partial y}$: Take $\frac{\partial}{\partial y}$ (z is a function of y)

$$x^2 + x\frac{\partial z}{\partial y} + 3z^2\frac{\partial z}{\partial y} = 0$$

$$(x+3z^2)\frac{\partial z}{\partial y} = -x^2$$

$$\frac{\partial z}{\partial y} = \frac{-x^2}{x+3z^2}$$

(10)



$$V = x^2 h$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

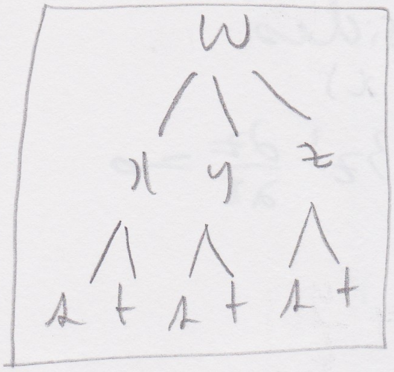
$$\frac{dV}{dt} = 2xh\frac{dx}{dt} + x^2\frac{dh}{dt}$$

$$\boxed{\text{Sub } x = 100\text{ cm} = h \quad \frac{dx}{dt} = 2\text{ cm/min} \quad \frac{dh}{dt} = -3\text{ cm/min}}$$

$$\frac{dV}{dt} = 20{,}000(2) + 10{,}000(-3)$$

$$= 10{,}000 \text{ cm}^3/\text{min} \quad \text{or } 0.01 \text{ m}^3/\text{min}$$

(11)



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (3x^2 - 6xy + z^2)(2s)$$
$$- 3x^2(2s)$$
$$+ 2xz(1)$$

when $s = 2$, $t = 1$:

$x = 5 \quad y = 3 \quad z = 3$

Sub in:

$$\frac{\partial w}{\partial s} = -6(4) - 75(4) + 30(1) = -294$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= (3x^2 - 6xy + z^2)(2t) \quad - 3x^2(-2t)$$
$$+ 2xz(1)$$

Sub in with values above:

$$= -6(2) - 75(-2) + 30(1)$$
$$= 168$$

(12) 1) Compute $D_{\vec{u}}\, T(\vec{x})$ (which gives $\frac{dT}{ds}$)

$$\vec{u} = \frac{[1,2,2]}{\|[1,2,2]\|} = \frac{1}{3}[1,2,2]$$

$$\nabla T(\vec{x}) = \left[\frac{y}{3} + \frac{z}{6},\ \frac{x}{3} + \frac{z}{6},\ \frac{x}{6} + \frac{y}{6}\right]$$

$$\nabla T(1,1,1) = \left[\frac{1}{2},\ \frac{1}{2},\ \frac{1}{3}\right]$$

$$D_{\vec{u}}\, T(1,1,1) = \nabla T(1,1,1) \cdot \vec{u}$$

$$= \frac{1}{3}\left[\frac{1}{2} + 1 + \frac{2}{3}\right] \leftarrow \text{gives } \frac{dT}{ds}$$

2) $$\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt} \leftarrow \text{speed}$$

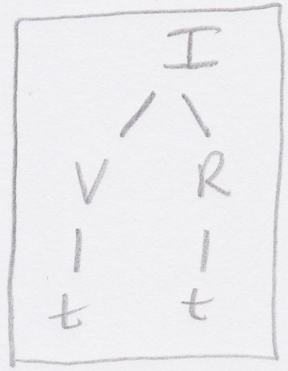
$$= \frac{1}{3}\left[\frac{1}{2} + 1 + \frac{2}{3}\right] 20 \quad \frac{{}^\circ C}{km} \cdot \frac{km}{h}$$

$$= \frac{20}{3}\left(\frac{1}{2} + 1 + \frac{2}{3}\right) {}^\circ C/h$$

$$\simeq 14.4\ {}^\circ C/h$$

(13)

$$I = VR^{-1}$$



$$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt}$$

$$= R^{-1}\frac{dV}{dt} - VR^{-2}\frac{dR}{dt}$$

$V = RI$

$$\frac{dI}{dt} = R^{-1}\frac{dV}{dt} - R^{-1}I\frac{dR}{dt}$$

Sub in:

$$\frac{dI}{dt} = (400)^{-1}(-0.01) - (400)^{-1}(0.08)(0.03)$$

$$= -3.1 \times 10^{-5} \text{ A/s} \quad \text{or} \quad -31 \ \mu\text{A/s}$$

(14) Let $\nabla f(x_0, y_0) = [a, b]$

$$D_{\bar{u}} f(x_0, y_0) = -1$$

$$\nabla f(x_0, y_0) \cdot \bar{u} = -1$$

$$[a, b] \cdot \left[\frac{-4}{5}, \frac{3}{5}\right] = -1$$

$$-\frac{4}{5}a + \frac{3}{5}b = -1 \qquad (1)$$

$$D_{\bar{v}} f(x_0, y_0) = 2$$

$$\nabla f(x_0, y_0) \cdot \vec{v} = 2$$

$$[a, b] \cdot \left[\frac{3}{5}, \frac{4}{5}\right] = 2$$

$$\frac{3}{5}a + \frac{4}{5}b = 2 \qquad (2)$$

3 × (1)
4 × (2) :

$$-\frac{12}{5}a + \frac{9}{5}b = -3$$

$$+ \quad \frac{12}{5}a + \frac{16}{5}b = 8$$

$$5b = 5$$

$$\Rightarrow b = 1$$

$b = 1 \rightarrow (2)$:

$$\frac{3}{5}a + \frac{4}{5} = 2$$

$$\frac{3}{5}a = \frac{6}{5}$$

$$a = 2$$

$$\nabla f(x_0, y_0) = [2, 1]$$

(15) a) $\bar{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{5}}[2,1]$

$\nabla T(x,y) = [-2x, -4y]$

$D_{\bar{u}} T(4,3) = \nabla T(4,3) \cdot \bar{u}$

$= [-8,-12] \cdot \frac{1}{\sqrt{5}}[2,1]$

$= \frac{-28}{\sqrt{5}}$

b) Direction of maximum increase at point $(4,3)$ is $\nabla T(4,3) = [-8,-12]$

Unit vector: $\frac{1}{\sqrt{208}}[-8,-12] = \frac{1}{\sqrt{13}}[-2,-3]$

(16) Normal vector

$\bar{n} = [-z_x, -z_y, 1]$

$= [-6x, -2y, 1]$

at $(1,2,7)$ $\bar{n} = [-6,-4,1]$

Tangent plane $-6x - 4y + z = d$

Sub $(1,2,7)$: $-6 - 8 + 7 = d$

$d = -7$

$\boxed{-6x - 4y + z = -7}$ or $\boxed{6x + 4y - z = 7}$

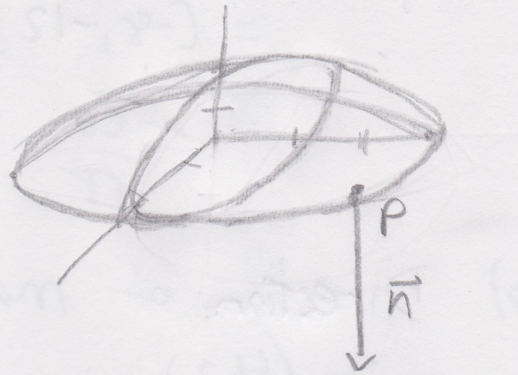
(17) $2x^2+y^2+3z^2=9$

$\nabla f(\vec{x})$ is normal to $f=c$

$\nabla f(\vec{x}) = [4x, 2y, 6z]$

$\nabla f(1,2,-1) = [4, 4, -6]$

$\vec{n} = [4, 4, -6]$



Normal Line:

$\vec{x} = \vec{p} + \vec{n}t$

$[x, y, z] = [1, 2, -1] + [4, 4, -6]t$

$$\begin{cases} x = 1+4t \\ y = 2+4t \\ z = -1-6t \end{cases}$$

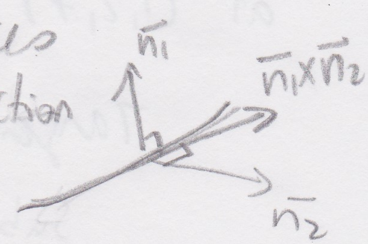
(18) $S_1: z = x^2+y^2$ $\qquad$ $S_2: xyz = 10$

Let $\vec{n}_1$ = normal to S_1 $\qquad$ $\vec{n}_2$ = normal to S_2

$\vec{n}_1 \times \vec{n}_2$ is $\perp$ to $\vec{n}_1$ and $\vec{n}_2$

$\Rightarrow$ $\vec{n}_1 \times \vec{n}_2$ is tangent to both surfaces

$\Rightarrow$ $\vec{n}_1 \times \vec{n}_2$ is tangent to curve of intersection



$\rightarrow$

(18) Cont'd

$\bar{n}_1 = [-z_x, -z_y, 1]$

$= [-2x, -2y, 1]$

$= [-2, -4, 1]$

$\bar{n}_2 = \nabla(xyz)$

$= [yz, xz, xy]$

$= [10, 5, 2]$

$\vec{n}_1 \times \vec{n}_2 = [-13, 14, 30]$

Alternatively: $[13, -14, -30]$

(19) Find critical points

$z_x = 8x^2 - 4x^3 = 4x^2(2-x) = 0 \Rightarrow x = 0, 2$

$z_y = 12y^2 - 4y^3 = 4y^2(3-y) = 0$
$\Rightarrow y = 0, 3$

x	y	z
0	0	0
0	3	27
2	0	16/3
2	3	97/3 ← Max z (highest point)

Highest point is $(2, 3, \frac{97}{3})$

(20) Lagrange Multipliers

Let the point be (x, y, z)

Minimize $\sqrt{(x-2)^2+(y-1)^2+z^2}$

It's equivalent to minimize d^2

$$\boxed{\begin{array}{l} \text{Minimize } f = (x-2)^2+(y-1)^2+z^2 \\ \text{Subject to : } \underbrace{2x+y-z}_{g}=8 \end{array}}$$

$$\nabla f = \lambda \nabla g$$

$$[2(x-2),\ 2(y-1),\ 2z] = \lambda[2, 1, -1]$$

$$\left.\begin{array}{l} 2(x-2) = 2\lambda \\ 2(y-1) = \lambda \\ 2z = -\lambda \end{array}\right\}$$

$$\lambda = x-2 = 2(y-1) = -2z$$

$$y-1=-z$$

$$\boxed{y=1-z}$$

$$\boxed{x=2-2z}$$

Both $\rightarrow$ $2x+y-z=8$

$$2(2-2z)+(1-z)-z=8$$

$$5-6z=8$$

$$z=-\tfrac{1}{2} \Rightarrow x=3, \quad y=\tfrac{3}{2}$$

Point is $\left(3, \frac{3}{2}, \frac{-1}{2}\right)$

(21) Lagrange Multipliers

Minimize $C = 6[2xy] + 4[2xz] + 2[2yz]$

$$\boxed{\begin{array}{l} \text{Minimize } C = 12xy + 8xz + 4yz \\ \text{Subject to } g = c \ : \ xyz = 6 \end{array}}$$

$\nabla C = \lambda \nabla g$

$[12y + 8z,\ 12x + 4z,\ 8x + 4y] = \lambda\,[yz,\ xz,\ xy]$

$12y + 8z = \lambda(yz) \Rightarrow \quad \lambda = \dfrac{12y + 8z}{yz} = \dfrac{12}{z} + \dfrac{8}{y}$

$12x + 4z = \lambda(xz) \Rightarrow \quad \lambda = \dfrac{12x + 4z}{xz} = \dfrac{12}{z} + \dfrac{4}{x}$

$8x + 4y = \lambda xy \Rightarrow \quad \lambda = \dfrac{8x + 4y}{xy} = \dfrac{8}{y} + \dfrac{4}{x}$

$\dfrac{12}{z} + \dfrac{8}{y} = \dfrac{12}{z} + \dfrac{4}{x} = \dfrac{8}{y} + \dfrac{4}{x}$

$\dfrac{8}{y} = \dfrac{4}{x} \Rightarrow \boxed{y = 2x}$ $\qquad \dfrac{12}{z} = \dfrac{4}{x} \Rightarrow \boxed{z = 3x}$

$\begin{cases} z = 3x \\ y = 2x \end{cases} \rightarrow \quad \begin{array}{l} xyz = 6 \\ 6x^3 = 6 \end{array}$

$\Rightarrow x = 1,\ y = 2,\ z = 3$ (m)

for a minimum cost of \$72

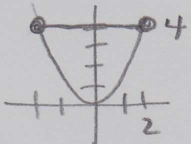
(22) 1) interior critical points

$f_x = 6x = 0$
$f_y = 4y - 4 = 0$ } both 0 or undefined?

$\Rightarrow (x,y) = (0,1)$

$f(0,1) = -2$

2) Evaluate f on boundary:
Corners and critical points on sides



Corner	$f = 3x^2 + 2y^2 - 4y$
$(2,4)$	28
$(-2,4)$	28

Side 1: $y = 4,\ -2 < x < 2$

$y = 4 \rightarrow f = 3x^2 + 2y^2 - 4y$
$f = 3x^2 + 16$

Set $f' = 0$: $6x = 0$
$\Rightarrow x = 0$

Point $(0,4)$ $f(0,4) = 16$

Side 2: $y = x^2,\ -2 < x < 2$

$y = x^2 \rightarrow f = 3x^2 + 2y^2 - 4y$
$f = 3x^2 + 2x^4 - 4x^2$
$f = 2x^4 - x^2$
$f' = 8x^3 - 2x = 0$
$2x(4x^2 - 1) = 0$
$2x(2x+1)(2x-1) = 0$
$\Rightarrow x = 0, -\frac{1}{2}\ \frac{1}{2}$

x	$y = x^2$	$f = 3x^2 + 2y^2 - 4y$
0	0	0
$-\frac{1}{2}$	1/4	$-\frac{1}{8}$
$\frac{1}{2}$	1/4	$-\frac{1}{8}$

↗

SUMMARY

Point	f	
$(0,1)$	-2	← MIN
$(\pm 2,4)$	28	← MAX
$(0,4)$	16	
$(0,0)$	0	
$(\pm\frac{1}{2}, \frac{1}{4})$	$-\frac{1}{8}$	

$\boxed{\text{SEE NEXT PAGE}}$ ⇒

(22) Cont'd

Absolute Minimum of $f = -2$

at $(x,y) = (0,1)$

Absolute Maximum of $f = 28$

at $(x,y) = (\pm 2, 4)$

(23) $f_x = 3x^2 + y^2 - 6x = 0$

$f_y = 2xy - 8y = 2y(x-4) = 0$

$3x^2 + y^2 - 6x = 0$

and

$x = 4$ or $y = 0$

Case 1 $\quad x = 4 \Rightarrow y^2 + 24 = 0$ $\quad$ no solution

Case 2 $\quad y = 0 \Rightarrow 3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0, 2$

Critical Points: $(0,0)$ and $(2,0)$

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x-6 & 2y \\ 2y & 2x-8 \end{vmatrix}$$

$\rightarrow$

(23) Cont'd

At $(0,0)$, $\Delta > 0$ and $f_{xx} < 0$
$\Rightarrow$ local max

At $(2,0)$, $\Delta = \begin{vmatrix} 6 & 0 \\ 0 & -4 \end{vmatrix} < 0 \Rightarrow$ saddle point

(24) $f_x = 3x^2 - 3y = 0 \Rightarrow x^2 = y$ and
$f_y = 2y - 3x = 0 \Rightarrow y = 3/2\, x$
$\Rightarrow x^2 = 3/2\, x \Rightarrow x(x - \frac{3}{2}) = 0$
$\Rightarrow x = 0, 3/2$

Critical points $(0,0)$, $(\frac{3}{2}, \frac{9}{4})$

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -3 \\ -3 & 2 \end{vmatrix}$$

At $(0,0)$ $\Delta = \begin{vmatrix} 0 & -3 \\ -3 & 2 \end{vmatrix} < 0 \Rightarrow$ saddle point

At $(\frac{3}{2}, \frac{9}{4})$ $\Delta = \begin{vmatrix} 9 & -3 \\ -3 & 2 \end{vmatrix} > 0$ and $f_{xx} > 0$
$\Rightarrow$ local min