

1.

Calculate the divergence and curl of $\mathbf{F} = [x^2 - yz, e^y - xz, z^3 - y^2]$.

$$\begin{aligned}\operatorname{div} \vec{F} &= \frac{\partial}{\partial x} (x^2 - yz) + \frac{\partial}{\partial y} (e^y - xz) + \frac{\partial}{\partial z} (z^3 - y^2) \\ &= 2x + e^y + 3z^2\end{aligned}$$

$$\begin{aligned}\operatorname{Curl} \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & e^y - xz & z^3 - y^2 \end{vmatrix} \\ &= \vec{i} [-2y + x] - \vec{j} [0 + y] \\ &\quad + \vec{k} [-z + z] \\ &= [x - 2y, -y, 0]\end{aligned}$$

2. Evaluate $\int_C (2x + 3z) ds$ where C is given by:

$$x = 3 \cos t, \quad y = 3 \sin t, \quad z = 4t, \quad 0 \leq t \leq 4\pi$$

$$\frac{dx}{dt} = -3 \sin t \quad \frac{dy}{dt} = 3 \cos t \quad \frac{dz}{dt} = 4$$

$$ds = \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + 4^2} dt$$

$$= \sqrt{9 \sin^2 t + 9 \cos^2 t + 4^2} dt$$

$$= \sqrt{9 + 16} dt$$

$$= \sqrt{25} dt$$

$$= 5 dt$$

$$\int_C (2x + 3z) ds = 5 \int_0^{4\pi} (6 \cos t + 12t) dt$$

$$= 5 [6 \sin t + 6t^2]_0^{4\pi}$$

$$= 5 [96 \pi^2]$$

$$= 480 \pi^2$$

3. Consider the conservative vector field $\mathbf{F} = [2x, 3y^2z, y^3 - 2z]$.

a) Find a potential for $\mathbf{F}$

$$\begin{aligned} f &= \int 2x dx \quad \text{AND} \quad f = \int 3y^2z dy \quad \text{AND} \quad f = \int (y^3 - 2z) dz \\ &= x^2 + g(y, z) \quad \quad \quad = y^3z + h(x, z) \quad \quad \quad = y^3z - z^2 + k(x, y) \end{aligned}$$

$$\Rightarrow f = x^2 + y^3z - z^2$$

b) Let C be any path from $(1, 2, 3)$ to $(4, 3, 5)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$

Because $\vec{F}$ is conservative:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}, \text{ where } f \text{ is a potential for } \vec{F}$$

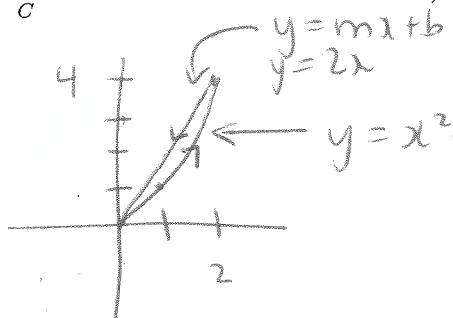
$$= f \Big|_A^B$$

$$= (x^2 + y^3z - z^2) \Big|_{(1, 2, 3)}^{(4, 3, 5)}$$

$$= 126 - 16$$

$$= 110$$

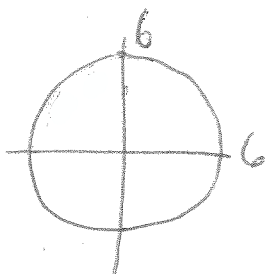
4. Let C be the part of $y = x^2$ from $(0, 0)$ to $(2, 4)$ followed by the straight line segment from $(2, 4)$ to $(0, 0)$. Use Green's Theorem to evaluate $\oint_C [\arctan e^x dx + (x^2 + \arcsin e^y) dy]$



$$R: \quad \begin{aligned} x^2 &\leq y \leq 2x \\ 0 &\leq x \leq 2 \end{aligned}$$

$$\begin{aligned} \oint (Pdx + Qdy) &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \int_0^2 \int_{x^2}^{2x} 2x \, dy \, dx \\ &= \int_0^2 2xy \Big|_{y=x^2}^{y=2x} dx \\ &= \int_0^2 (4x^2 - 2x^3) dx \\ &= \left[\frac{4x^3}{3} - \frac{x^4}{2} \right]_0^2 \\ &= \frac{32}{3} - 8 \\ &= \frac{8}{3} \end{aligned}$$

5. Use the 2D Divergence Theorem to calculate the flux of $\mathbf{F} = [xy^2, x^2y + \ln(\sin x)]$ across the circle of radius 6 centred at the origin.



Use polar

$$R: \begin{array}{l} 0 \leq r \leq 6 \\ 0 \leq \theta \leq 2\pi \end{array}$$

$$\begin{aligned} \Phi &= \iint_R (\operatorname{div} \bar{\mathbf{F}}) dA \\ &= \iint_R \underbrace{(y^2 + x^2)}_{r^2} \underbrace{dA}_{r dr d\theta} \end{aligned}$$

$$= \int_0^{2\pi} \int_0^6 r^3 dr d\theta$$

$$= \int_0^{2\pi} \left. \frac{r^4}{4} \right|_0^6 d\theta$$

$$= \frac{6^4}{4} \int_0^{2\pi} d\theta$$

$$= 648\pi$$