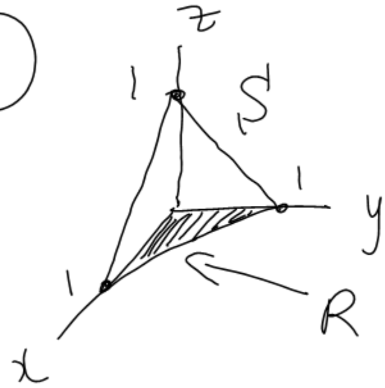
①

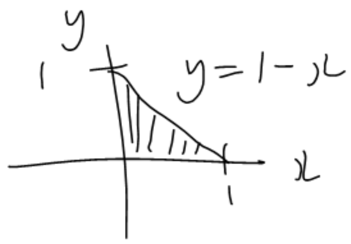


$$x + y + z = 1$$
$$z = 1 - x - y$$

$$z_x = -1$$
$$z_y = -1$$

$$dS = \sqrt{1 + (z_x)^2 + (z_y)^2}\, dy\, dx$$
$$= \sqrt{3}\, dy\, dx$$

R :



$$0 \le y \le 1 - x$$
$$0 \le x \le 1$$

$$\iint_S f\, dS = \int_0^1 \int_0^{1-x} (x+y)\sqrt{3}\, dy\, dx$$

$$= \sqrt{3} \int_0^1 \int_0^{1-x} (x+y)\, dy\, dx$$

$\longrightarrow$

① Cont'd

$$= \sqrt{3} \int_0^1 \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=1-x} dx$$

$$= \sqrt{3} \int_0^1 \Big[\underbrace{x(1-x) + \frac{1}{2}(1-x)^2}_{} \Big] dx$$

$$x - x^2 + \frac{1}{2}(1 - 2x + x^2)$$

$$= x - x^2 + \frac{1}{2} - x + \frac{x^2}{2}$$

$$= -\frac{x^2}{2} + \frac{1}{2}$$

$$= \sqrt{3} \int_0^1 \left[-\frac{x^2}{2} + \frac{1}{2} \right] dx$$

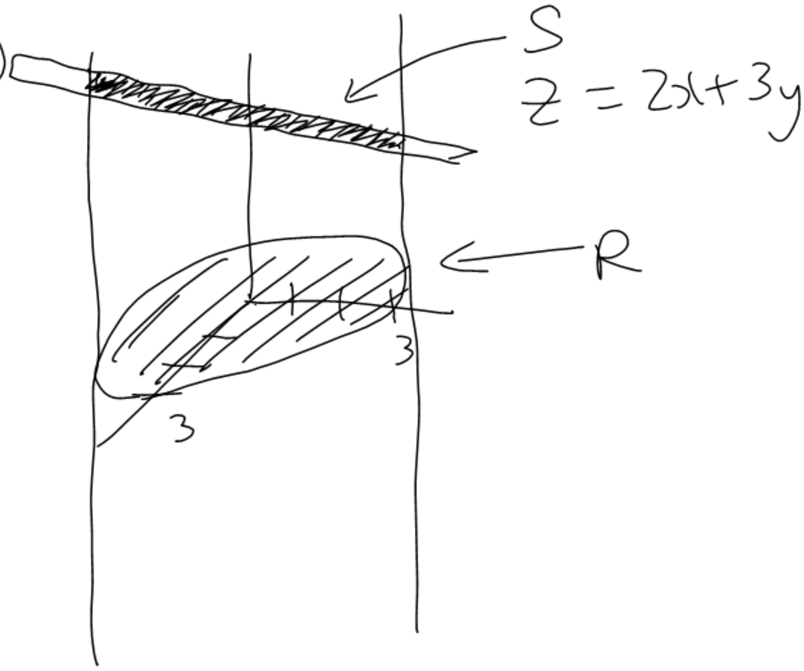
$$= \sqrt{3} \left[\frac{-x^3}{6} + \frac{x}{2} \right]_0^1$$

$$= \sqrt{3} \left[\frac{-1}{6} + \frac{1}{2} \right]$$

$$= \sqrt{3} \left[\frac{1}{3} \right]$$

$$= \frac{\sqrt{3}}{3}$$

③



$$z_x = 2, \quad z_y = 3$$

$$dS = \sqrt{1+(z_x)^2+(z_y)^2}\, dy\,dx$$

$$= \sqrt{14}\, dy\,dx$$

$$= \sqrt{14}\, \underline{\underline{r}}\, dr\, d\theta$$

$$\boxed{\begin{aligned} R:\quad & x^2+y^2=9 \\ & r^2 = 9 \\ & r = \pm 3 \\ & r = 3 \\ & 0 \le r \le 3 \\ & 0 \le \theta \le 2\pi \end{aligned}}$$

$\longrightarrow$

③ Cont'd The integrand is $f = y + z + 3$

$\boxed{\text{Recall } z = 2x + 3y}$

$= y + 2x + 3y + 3$

$= 2x + 4y + 3$

$\boxed{\begin{array}{l}\text{Recall } x = r\cos\theta \\ \qquad\quad\; y = r\sin\theta\end{array}}$

$= 2r\cos\theta + 4r\sin\theta + 3$

$$\iint_S f\,dS = \int_0^{2\pi}\int_0^3 (2r\cos\theta + 4r\sin\theta + 3)\sqrt{14}\,r\,dr\,d\theta$$

$$= \sqrt{14}\int_0^{2\pi}\int_0^3 (2r^2\cos\theta + 4r^2\sin\theta + 3r)\,dr\,d\theta$$

$$= \sqrt{14}\int_0^{2\pi}\left[\frac{2r^3}{3}\cos\theta + \frac{4r^3}{3}\sin\theta + \frac{3r^2}{2}\right]_{r=0}^{r=3} d\theta$$

$$= \sqrt{14}\int_0^{2\pi}\left[18\cos\theta + 36\sin\theta + \frac{27}{2}\right] d\theta$$

$$= \sqrt{14}\left[18\sin\theta - 36\cos\theta + \frac{27}{2}\theta\right]_0^{2\pi}$$

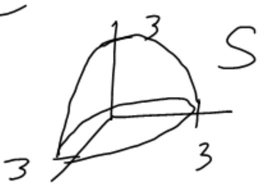
$\rightarrow$

③ Cont'd $= \sqrt{14}\left[(-36+27\pi)-(-36)\right]$

$= \sqrt{14}\left[27\pi\right]$

$= 27\sqrt{14}\,\pi$

(13) S is the upper hemisphere



$$z = \sqrt{9 - x^2 - y^2}$$

$$z^2 = 9 - x^2 - y^2$$

$$x^2 + y^2 + z^2 = 9$$

$\Longrightarrow$ radius $= 3$

Spherical $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$,
$z = \rho \cos\phi$ and we know $\rho = 3$.

Position vector is :

$$\vec{r} = [3\sin\phi\cos\theta,\ 3\sin\phi\sin\theta,\ 3\cos\phi]$$

$R: 0 \le \phi \le \frac{\pi}{2}$ (<u>upper</u> hemisphere), $0 \le \theta \le 2\pi$

$$\vec{r}_\phi = [3\cos\phi\cos\theta,\ 3\cos\phi\sin\theta,\ -3\sin\phi]$$

$$\vec{r}_\theta = [-3\sin\phi\sin\theta,\ 3\sin\phi\cos\theta,\ 0]$$

$\longrightarrow$

(13) Cont'd

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3\cos\phi\cos\theta & 3\cos\phi\sin\theta & -3\sin\phi \\ -3\sin\phi\sin\theta & 3\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \vec{i}\left[9\sin^2\phi\cos\theta\right] - \vec{j}\left[-9\sin^2\phi\sin\theta\right]$$
$$+ \vec{k}\left[9\cos\phi\sin\phi(\cos^2\theta + \sin^2\theta)\right]$$
$$= \left[9\sin^2\phi\cos\theta,\ 9\sin^2\phi\sin\theta,\ 9\cos\phi\sin\phi\right]$$

$$\vec{n}\,dS = \pm(\vec{r}_\phi \times \vec{r}_\theta)\,d\phi\,d\theta$$

Upwards $\Rightarrow$ z-component > 0
$9\cos\phi\sin\phi$ is > 0 for $0 < \phi < \frac{\pi}{2}$
So choose (+)

$$\vec{n}\,dS = (\vec{r}_\phi \times \vec{r}_\theta)\,d\phi\,d\theta$$

$$\vec{F} = [x, y, 0]$$

From position vector, $x = 3\sin\phi\cos\theta$
$y = 3\sin\phi\sin\theta$

$$\vec{F} = [3\sin\phi\cos\theta,\ 3\sin\phi\sin\theta,\ 0]$$

$$\vec{F}\cdot\vec{n}\,dS = [3\sin\phi\cos\theta,\ 3\sin\phi\sin\theta,\ 0]\cdot [9\sin^2\phi\cos\theta,\ 9\sin^2\phi\sin\theta,\ 9\cos\phi\sin\phi]\,d\phi\,d\theta$$

$\longrightarrow$

13 Cont'd

$$\vec{F} \cdot \vec{n}\, dS = 27 \sin^3\phi (\cos^2\theta + \sin^2\theta)\, d\phi\, d\theta$$

$$= 27 \sin^3\phi \, d\phi\, d\theta$$

$$\Phi = \iint_S \vec{F} \cdot \vec{n}\, dS$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 27 \underbrace{\sin^3\phi}_{\sin^2\phi \sin\phi} \, d\phi\, d\theta$$

$$\sin^2\phi \sin\phi$$
$$= (1-\cos^2\phi)\sin\phi$$
$$= \sin\phi - \cos^2\phi \sin\phi$$

$$= 27 \int_0^{2\pi}\int_0^{\pi/2} [\sin\phi - \cos^2\phi \sin\phi]\, d\phi\, d\theta$$

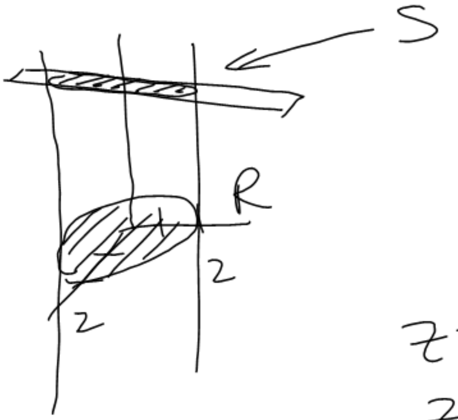
$$= 27 \int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3} \right]_{\phi=0}^{\phi=\frac{\pi}{2}} d\theta$$

$$= 27 \int_0^{2\pi} [\, 0 - (-1 + \tfrac{1}{3})\,]\, d\theta$$

$$= 27 \left(\tfrac{2}{3}\right) \theta \Big|_0^{2\pi}$$

$$= 36\pi$$

15



R: use polar

$x^2+y^2=4$

$r^2=4$

$r=\pm 2$

$r=2$

$0\le r\le 2,\ 0\le\theta\le 2\pi$

$z=3x+2$

$z_x=3$

$z_y=0$

$\vec{n}\,dS=\pm[-z_x,-z_y,1]\,dy\,dx$

$=\pm[-3,0,1]\,dy\,dx$

Upwards $\Rightarrow$ z-component >0

1 is >0 so choose $+$

$\vec{n}\,dS=[-3,0,1]\,dy\,dx$

$\vec{F}=[0,2y,3z]$

But $z=3x+2$

$=[0,2y,9x+6]$

$\vec{F}\cdot\vec{n}\,dS=[0,2y,9x+6]\cdot[-3,0,1]\,dy\,dx$

$=(9x+6)\,dy\,dx$

We'll convert this to polar.

$\longrightarrow$

(15) Cont'd

In polar, $x = r\cos\theta$
$dy\,dx = r\,dr\,d\theta$

$$\vec{F}\cdot\vec{n}\,dS = (9x+6)\,dy\,dx$$
$$= (9r\cos\theta + 6)\,r\,dr\,d\theta$$

$$\Phi = \iint_S \vec{F}\cdot\vec{n}\,dS$$
$$= \int_0^{2\pi}\int_0^{2} (9r\cos\theta + 6)\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^{2} (9r^2\cos\theta + 6r)\,dr\,d\theta$$
$$= \int_0^{2\pi} \Big[3r^3\cos\theta + 3r^2\Big]_{r=0}^{r=2}\,d\theta$$
$$= \int_0^{2\pi} (24\cos\theta + 12)\,d\theta$$
$$= \Big[24\sin\theta + 12\theta\Big]_0^{2\pi}$$
$$= 24\pi$$

(37) Set up the integral, in terms of a and b, for I_z. Given $\delta = 1$.

Given $x = au\cos v$, $y = bu\sin v$,
$z = u^2$, $0 \le u \le c$, $0 \le v \le 2\pi$.

In other words, the position vector is $\vec{r} = [au\cos v, bu\sin v, u^2]$
and R: $0 \le u \le c$, $0 \le v \le 2\pi$.

$$\vec{r}_u = [a\cos v, b\sin v, 2u]$$

$$\vec{r}_v = [-au\sin v, bu\cos v, 0]$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos v & b\sin v & 2u \\ -au\sin v & bu\cos v & 0 \end{vmatrix}$$

$$= [-2bu^2\cos v, -2au^2\sin v, abu(\cos^2 v + \sin^2 v)]$$

$$= [-2bu^2\cos v, -2au^2\sin v, abu]$$

$\rightarrow$

(37) Cont'd

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4b^2u^4\cos^2 v + 4a^2u^4\sin^2 v + a^2b^2u^2}$$

Does not simplify easily.

$$\text{Now } I_z = \iint_S (x^2+y^2)\,\delta\, dS$$

1 $\|\vec{r}_u \times \vec{r}_v\|\,du\,dv$

$x = au\cos v$
$y = bu\sin v$

$$I_z = \int_0^{2\pi}\int_0^{c} \left(a^2u^2\cos^2 v + b^2u^2\sin^2 v\right)\sqrt{4b^2u^4\cos^2 v + 4a^2u^4\sin^2 v + a^2b^2u^2}\,du\,dv$$

Do not evaluate!

Geometry Note: S is an elliptic paraboloid (cross-sections parallel to xy-plane are ellipses).