① $f = x^2 + y^2$, $x = 4t-1$, $y = 3t+1$, $-1 \le t \le 1$

a) $\dfrac{dx}{dt} = 4 \qquad \dfrac{dy}{dt} = 3$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \sqrt{16 + 9}\, dt$$

$$= \sqrt{25}\, dt$$

$$= 5\, dt$$

$$\int_C f\, ds = 5 \int_C (x^2 + y^2)\, dt$$

$$= 5 \int_{-1}^{1} \left[(4t-1)^2 + (3t+1)^2\right] dt$$

$$= 5 \int_{-1}^{1} \left[16t^2 - 8t + 1 + 9t^2 + 6t + 1\right] dt$$

$$= 5 \int_{-1}^{1} \left[25t^2 - 2t + 2\right] dt$$

$$= 5 \left[\frac{25t^3}{3} - t^2 + 2t\right]_{-1}^{1}$$

$$= 5 \left[\left(\frac{25}{3} - 1 + 2\right) - \left(-\frac{25}{3} - 1 - 2\right)\right]$$

$\rightarrow$

① Cont'd

$$= 5\left[\frac{62}{3}\right]$$

$$= \frac{310}{3}$$

b) $\int_C f\,dx = \int_C (x^2+y^2)\,dx$

From part a), $x^2+y^2 = 25t^2-2t+2$
and $\frac{dx}{dt}=4 \Rightarrow dx = 4dt$

$$= 4\int_{-1}^{1}(25t^2-2t+2)\,dt$$

$$= 4\left[\frac{25t^3}{3} - t^2 + 2t\right]_{-1}^{1}$$

$$= 4\left[\left(\frac{25}{3}-1+2\right)-\left(\frac{-25}{3}-1-2\right)\right]$$

$$= 4\left[\frac{62}{3}\right]$$

$$= \frac{248}{3}$$

c) $\int_C f\,dy = \int_C (x^2+y^2)\,dy$

From part a), $x^2+y^2 = 25t^2-2t+2$
and $\frac{dy}{dt}=3 \Rightarrow dy = 3dt$

$\longrightarrow$

① Cont'd

$$= 3\int_{-1}^{1} [25t^2 - 2t + 2]\, dt$$

$$= 3 \left[\frac{25t^3}{3} - t^2 + 2t \right]_{-1}^{1}$$

$$= 3 \left[\left(\frac{25}{3} - 1 + 2 \right) - \left(\frac{-25}{3} - 1 - 2 \right) \right]$$

$$= 3 \left[\frac{62}{3} \right]$$

$$= 62$$

⑤ $f = xy$, $x = 3t$, $y = t^4$, $0 \le t \le 1$

a) $\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = 4t^3$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \sqrt{9 + 16t^6}\,dt$$

$$\int_C f\,ds = \int_C xy\,ds$$

$$= \int_0^1 3t\,(t^4)\sqrt{9+16t^6}\,dt$$

$$= 3\int_0^1 t^5\sqrt{9+16t^6}\,dt$$

$$\boxed{\begin{aligned} &\text{Let } u = 9+16t^6 \\ &du = 96t^5\,dt \\ &\frac{du}{96} = t^5\,dt \\ &t=0 \Rightarrow u=9 \\ &t=1 \Rightarrow u=25 \end{aligned}}$$

$$= \frac{3}{96}\int_9^{25} \sqrt{u}\,du$$

$$= \frac{3}{96}\left[\frac{2}{3}u^{3/2}\right]_9^{25} \quad \rightarrow$$

(5) Cont'd

$$= \frac{2}{96} u^{3/2} \Big|_{9}^{25}$$

$$= \frac{2}{96} [125 - 27]$$

$$= \frac{196}{96}$$

$$= \frac{49}{24}$$

b) $$\int_C f\,dx = \int_C xy\,dx$$

$$= \int_0^1 3t^5 (3\,dt)$$

$$= 9 \int_0^1 t^5\,dt$$

$$= \frac{9t^6}{6} \Big|_0^1$$

$$= \frac{9}{6}$$

$$= \frac{3}{2}$$

c) $$\int_C f\,dy = \int_C xy\,dy$$

$$= \int_0^1 3t^5 (4t^3\,dt)$$

$\longrightarrow$

5 Cont'd

$$= 12 \int_0^1 t^8 \, dt$$

$$= \frac{12t^9}{9} \Big|_0^1$$

$$= \frac{12}{9}$$

$$= \frac{4}{3}$$

7 $P = y^2, \quad Q = x$

$C: \ x = y^3$ from $(-1,-1)$ to $(1,1)$

Parametrize C by $y = t, \ x = t^3$

$-1 \leq t \leq 1$

$dy = dt, \quad dx = 3t^2 dt$

$$\int_C (P\,dx + Q\,dy) = \int_C (y^2 dx + x\,dy)$$

$$= \int_{-1}^{1} \left[t^2 (3t^2 dt) + t^3 dt \right]$$

$$= \int_{-1}^{1} [3t^4 + t^3] dt$$

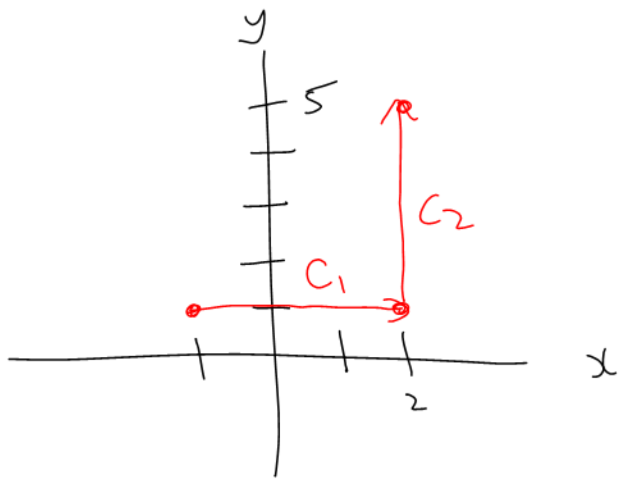
$$= \left[\frac{3t^5}{5} + \frac{t^4}{4} \right]_{-1}^{1}$$

$$= \left[\frac{3}{5} + \frac{1}{4} - \left(\frac{-3}{5} + \frac{1}{4} \right) \right]$$

$$= \frac{6}{5}$$

(9) $P = x^2 y, \quad Q = xy^3$

C: line segments from $(-1,1)$ to $(2,1)$ and $(2,1)$ to $(2,5)$



Parametrize C_1 and C_2.

C_1: $x = t, \quad y = 1, \quad -1 \le t \le 2$
$dx = dt, \quad dy = 0$

C_2: $x = 2, \quad y = t, \quad 1 \le t \le 5$
$dx = 0 \quad dy = dt$

ALTERNATIVELY

Recall that parametrizations are not unique. Another correct way:

C_1: $x = -1 + 3t, \quad y = 1, \quad 0 \le t \le 1$
$dx = 3\,dt, \quad dy = 0$

C_2: $x = 2, \quad y = 1 + 4t, \quad 0 \le t \le 1$
$dx = 0, \quad dy = 4\,dt$

$\longrightarrow$

⑨ Cont'd

$$\int_{C_1} (P\,dx + Q\,dy) = \int_{C_1} (x^2 y\,dx + xy^3\,dy)$$

$$= \int_{-1}^{2} t^2\,dt$$

$$= \left. \frac{t^3}{3} \right|_{-1}^{2}$$

$$= \frac{8}{3} - \left(\frac{-1}{3}\right)$$

$$= 3$$

$$\int_{C_2} (P\,dx + Q\,dy) = \int_{C_2} (x^2 y\,dx + xy^3\,dy)$$

$$= \int_{1}^{5} 2t^3\,dt$$

$$= \left. \frac{t^4}{2} \right|_{1}^{5}$$

$$= \frac{625}{2} - \frac{1}{2}$$

$$= 312$$

$$\int_{C} (P\,dx + Q\,dy) = 3 + 312$$

$$= 315$$

If you used the alternative parametrization you still get 315 as the final answer.

(13) $\vec{F} = [y, -x, z]$

$x = \sin t, \quad y = \cos t, \quad z = 2t, \quad 0 \le t \le \pi$

$dx = \cos t\,dt, \quad dy = -\sin t\,dt, \quad dz = 2dt$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C (P\,dx + Q\,dy + R\,dz)$$

$$= \int_C (y\,dx - x\,dy + z\,dz)$$

$$= \int_0^{\pi} (\cos^2 t\,dt + \sin^2 t\,dt + 4t\,dt)$$

$$= \int_0^{\pi} [\cos^2 t + \sin^2 t + 4t]\,dt$$

$$= \int_0^{\pi} [1 + 4t]\,dt$$

$$= [t + 2t^2]_0^{\pi}$$

$$= \pi + 2\pi^2$$

(17) $f = 2x + 9xy$

$C: \quad x = t, \quad y = t^2, \quad z = t^3, \quad 0 \le t \le 1$

$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2t, \quad \frac{dz}{dt} = 3t^2$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \sqrt{1 + 4t^2 + 9t^4}\, dt$$

$$\int_C f\, ds = \int_C (2x + 9xy)\, ds$$

$$= \int_0^1 [2t + 9t^3] \sqrt{1 + 4t^2 + 9t^4}\, dt$$

Sub $u = 1 + 4t^2 + 9t^4$

$du = (8t + 36t^3)\, dt$

$\frac{du}{4} = (2t + 9t^3)\, dt$

$t = 0 \Rightarrow u = 1$

$t = 1 \Rightarrow u = 14$

$$= \frac{1}{4} \int_1^{14} \sqrt{u}\, du$$

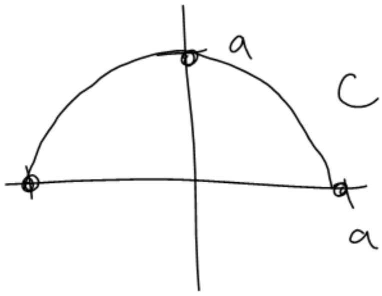
$$= \frac{1}{4} \left[\frac{2}{3} u^{3/2} \right]_1^{14} \longrightarrow$$

17 Cnt'd

$$= \frac{1}{6}\left[u^{3/2}\right]_1^{14}$$

$$= \frac{14\sqrt{14} - 1}{6}$$

(19)



To parametrize part of a circle, use $x = r\cos t$, $y = r\sin t$ for appropriate values of t and r = radius.

$$\boxed{\begin{array}{l} C: \ x = a\cos t, \ y = a\sin t, \quad 0 \le t \le \pi \\ \frac{dx}{dt} = -a\sin t, \quad \frac{dy}{dt} = a\cos t \end{array}}$$

Let $\delta = 1$ because the wire has uniform density.

By symmetry of C and δ, $\bar{x} = 0$.

$$m = \int_C \delta \, ds$$

$$= \boxed{\int_C ds} \leftarrow \text{Arc length of a semi-circle of radius } a$$

$$= a\pi$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \sqrt{a^2\sin^2 t + a^2\cos^2 t} \, dt$$

$\longrightarrow$

(19) Cont'd

$$= \sqrt{a^2(\sin^2 t + \cos^2 t)}\, dt$$

$$= |a|\, dt$$

$$= a\, dt \quad \text{because } a > 0.$$

$$\bar{y} = \frac{1}{m} \int_C y\, \delta\, ds$$

$$= \frac{1}{a\pi} \int_0^{\pi} a \sin t\, (1)\, a\, dt$$

$$= \frac{a}{\pi} \int_0^{\pi} \sin t\, dt$$

$$= \frac{a}{\pi} \left[-\cos t \right]_0^{\pi}$$

$$= \frac{a}{\pi} \left[1 - (-1) \right]$$

$$= \frac{2a}{\pi}$$

The centroid is $(\bar{x}, \bar{y}) = \left(0, \frac{2a}{\pi}\right)$.

(33) $\vec{F} = \dfrac{k}{x^2+y^2}(x\vec{i} + y\vec{j})$

$= \dfrac{k}{x^2+y^2}[x, y]$

a) C: line from $(1,0)$ to $(1,1)$

$x = 1, \quad y = t, \quad 0 \le t \le 1$

$dx = 0, \quad dy = dt$

$W = \int_C (P\,dx + Q\,dy)$

$= \int_C \left[\dfrac{kx}{x^2+y^2}\,dx + \dfrac{ky}{x^2+y^2}\,dy\right]$

$= k\int_0^1 \dfrac{t\,dt}{1+t^2}$

Let $u = 1+t^2$
$du = 2t\,dt$
$\dfrac{du}{2} = t\,dt$
$t=0 \Rightarrow u=1$
$t=1 \Rightarrow u=2$

$= \dfrac{k}{2}\int_1^2 \dfrac{du}{u}$

$= \dfrac{k}{2}\ln|u| \Big|_1^2$

$= \dfrac{k}{2}\ln 2$

There are other correct parametrizations of C. They all give the same final answer.

$\longrightarrow$

33 Cont'd

b) $\vec{F} = \frac{k}{x^2+y^2}[x, y]$

C: Line from $(1,1)$ to $(0,1)$

$x = 1-t, \quad y = 1, \quad 0 \le t \le 1$

$dx = -dt, \quad dy = 0$

$$W = \int_C (P\,dx + Q\,dy)$$

$$= \int_C \left[\frac{kx}{x^2+y^2}\,dx + \frac{ky}{x^2+y^2}\,dy\right]$$

$$= k\int_0^1 \frac{(1-t)}{(1-t)^2+1}(-dt)$$

$$= k\int_0^1 \frac{t-1}{t^2-2t+2}\,dt$$

Let $u = t^2-2t+2$

$du = (2t-2)\,dt$

$\frac{du}{2} = (t-1)\,dt$

$t=0 \Rightarrow u=2$

$t=1 \Rightarrow u=1$

$$= \frac{k}{2}\int_2^1 \frac{du}{u}$$

$$= \frac{k}{2}\ln|u|\Big|_2^1$$

$$= -\frac{k}{2}\ln 2$$

There are other correct parametrizations of C. They all give the same final answer.