① $\vec{F}(x,y) = \vec{i} + \vec{j}$

or $\vec{F}(x,y) = [1,1]$

Calculate a few vectors:

$\vec{F}(0,0) = [1,1]$

$\vec{F}(2,2) = [1,1]$

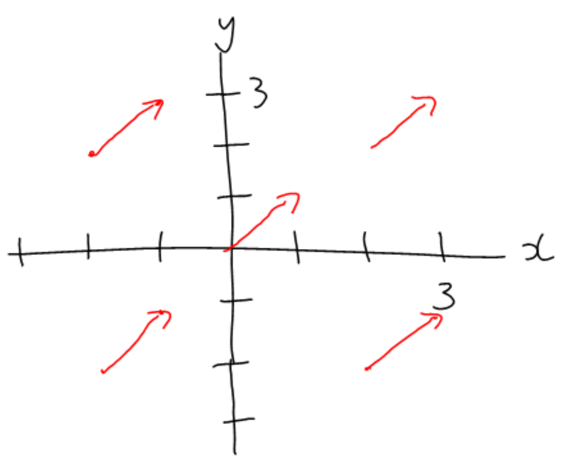
$\vec{F}(-2,2) = [1,1]$

$\vec{F}(-2,-2) = [1,1]$

$\vec{F}(2,-2) = [1,1]$

etc.

sketch:



(3) $\vec{F}(x,y) = x\vec{i} - y\vec{j}$

or $\vec{F}(x,y) = [x, -y]$

Calculate a few vectors:

$\vec{F}(1,1) = [1, -1]$

$\vec{F}(-1,1) = [-1, -1]$

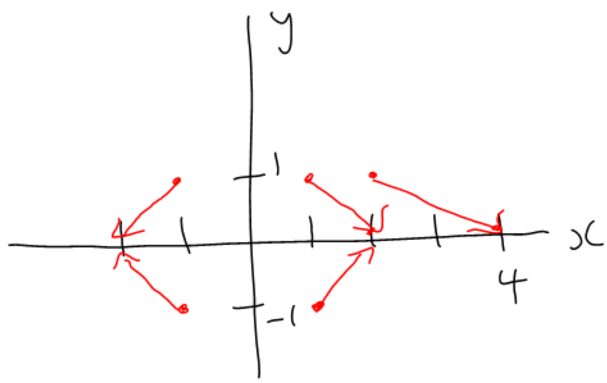
$\vec{F}(-1,-1) = [-1, 1]$

$\vec{F}(1,-1) = [1, 1]$

$\vec{F}(2,1) = [2, -1]$

etc.

Sketch:



(9) $\vec{F}(x,y,z) = -x\vec{i} - y\vec{j}$ or $\vec{F}(x,y,z) = -x\vec{i} - y\vec{j} + 0\vec{k}$

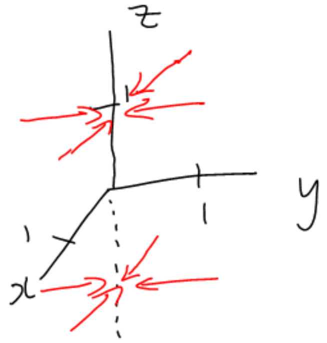
or $\vec{F}(x,y,z) = [-x, -y, 0]$

Calculate a few vectors:

$$\vec{F}(0,1,1) = [0,-1,0]$$
$$\vec{F}(0,-1,1) = [0,1,0]$$
$$\vec{F}(1,0,1) = [-1,0,0]$$
$$\vec{F}(-1,0,1) = [1,0,0]$$
$$\vec{F}(0,1,-1) = [0,-1,0]$$
$$\vec{F}(0,-1,-1) = [0,1,0]$$
$$\vec{F}(1,0,-1) = [-1,0,0]$$
$$\vec{F}(-1,0\ -1) = [1,0,0]$$

etc.

Sketch:



(19) $\vec{F}(x,y,z) = [xy^2, yz^2, zx^2]$

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F}$$
$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(zx^2)$$
$$= y^2 + z^2 + x^2$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & yz^2 & zx^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(zx^2) - \frac{\partial}{\partial z}(yz^2)\right]$$
$$- \vec{j}\left[\frac{\partial}{\partial x}(zx^2) - \frac{\partial}{\partial z}(xy^2)\right]$$
$$+ \vec{k}\left[\frac{\partial}{\partial x}(yz^2) - \frac{\partial}{\partial y}(xy^2)\right]$$
$$= \vec{i}[-2yz] - \vec{j}[2xz] + \vec{k}[-2xy]$$
$$= [-2yz, -2xz, -2xy]$$

(21) $\vec{F}(x,y,z) = (y^2+z^2)\vec{i} + (x^2+z^2)\vec{j} + (x^2+y^2)\vec{k}$

or $\vec{F}(x,y,z) = [y^2+z^2,\ x^2+z^2,\ x^2+y^2]$

$$\begin{aligned}
\operatorname{div}\vec{F} &= \nabla\cdot\vec{F} \\
&= \frac{\partial}{\partial x}(y^2+z^2) + \frac{\partial}{\partial y}(x^2+z^2) + \frac{\partial}{\partial z}(x^2+y^2) \\
&= 0+0+0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl}\vec{F} &= \nabla\times\vec{F} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} \\
&= \vec{i}\left[\frac{\partial}{\partial y}(x^2+y^2) - \frac{\partial}{\partial z}(x^2+z^2)\right] \\
&\quad - \vec{j}\left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial z}(y^2+z^2)\right] \\
&\quad + \vec{k}\left[\frac{\partial}{\partial x}(x^2+z^2) - \frac{\partial}{\partial y}(y^2+z^2)\right]
\end{aligned}$$

Cont'd

(21) Cont'd

$$= \vec{i}\,[2y - 2z]$$
$$- \vec{j}\,[2x - 2z]$$
$$+ \vec{k}\,[2x - 2y]$$

$$= [2y - 2z,\ 2z - 2x,\ 2x - 2y]$$

(23) $\vec{F}(x,y,z) = [x+\sin yz,\ y+\sin xz,\ z+\sin xy]$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \frac{\partial}{\partial x}(x+\sin yz) + \frac{\partial}{\partial y}(y+\sin xz) + \frac{\partial}{\partial z}(z+\sin xy)$$

$$= 1+1+1$$

$$= 3$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+\sin yz & y+\sin xz & z+\sin xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z+\sin xy) - \frac{\partial}{\partial z}(y+\sin xz)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(z+\sin xy) - \frac{\partial}{\partial z}(x+\sin yz)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(y+\sin xz) - \frac{\partial}{\partial y}(x+\sin yz)\right]$$

$\longrightarrow$ Cont'd

(23) Cont'd

$$= \vec{i}\left[x\cos xy - x\cos xz\right]$$
$$- \vec{j}\left[y\cos xy - y\cos yz\right]$$
$$+ \vec{k}\left[z\cos xz - z\cos yz\right]$$

$$= \left[x\cos xy - x\cos xz,\ y\cos yz - y\cos xy,\ z\cos xz - z\cos yz\right]$$

(32) Let $\vec{F} = [P, Q, R]$.

Show that $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] - \vec{j}\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right]$$

$$+ \vec{k}\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]$$

$$= \left[R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\right]$$

$$\operatorname{div}(\operatorname{curl} \vec{F}) = \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(P_z - R_x)$$

$$+ \frac{\partial}{\partial z}(Q_x - P_y)$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy}$$

$$+ Q_{xz} - P_{yz}$$

Recall: $R_{yx} = R_{xy}$ if R_{yx} and R_{xy} are both continuous.

Similarly, $Q_{zx} = Q_{xz}$ etc.

→

32 Cont'd

$$= R_{xy} - Q_{xz} + P_{yz} - R_{xy} + Q_{xz} - P_{yz}$$

$$= 0$$