(7) R is bounded by $x+y=1$, $x+y=2$, $2x-3y=2$, and $2x-3y=5$.

Find $A = \iint_R 1 \, dx\,dy$.

Let $u = x+y$ and $v = 2x-3y$.

$R: \quad 1 \le u \le 2, \quad 2 \le v \le 5.$

Integrand $= 1$

$$J_{(u,v)\to(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= -5$$

$$J_{(x,y)\to(u,v)} = \frac{1}{J_{(u,v)\to(x,y)}}$$

$$= -\frac{1}{5}$$

$$\left| J_{(x,y)\to(u,v)} \right| = \frac{1}{5}$$

$$A = \iint_R 1 \, dx\,dy$$

$$= \iint_R 1 \cdot \left| J_{(x,y)\to(u,v)} \right| du\,dv$$

Continued $\longrightarrow$

7 Cont'd

$$= \int_2^5 \int_1^2 \frac{1}{5}\, du\, dv$$

$$= \int_2^5 \frac{u}{5} \Big|_{u=1}^{u=2} dv$$

$$= \int_2^5 \frac{1}{5}\, dv$$

$$= \frac{v}{5} \Big|_2^5$$

$$= \frac{3}{5}$$

(9) Find the area of the first-quadrant region bounded by $xy=2$, $xy=4$, $xy^3=3$, and $xy^3=6$.

Let $u=xy$ and $v=xy^3$

$R: \; 2 \le u \le 4, \; 3 \le v \le 6.$

The integral is $A = \iint_R 1 \, dx\,dy$

so integrand $=1$.

$$J_{(u,v)\to(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} y & x \\ y^3 & 3xy^2 \end{vmatrix}$$

$$= 3xy^3 - xy^3$$

$$= 2xy^3$$

$$= 2v$$

$$J_{(x,y)\to(u,v)} = \frac{1}{2v} = \frac{1}{2}\frac{1}{v}$$

$$|J_{(x,y)\to(u,v)}| = \frac{1}{2}\frac{1}{v}$$ since $v>0$ (see R above).

Continued

⑨ Cont'd

$$A = \iint_R 1 \, dx\,dy$$

$$= \iint_R 1 \cdot |J_{(x,y)\to(u,v)}| \, du\,dv$$

$$= \int_3^6 \int_2^4 \frac{1}{2} \frac{1}{v} \, du\,dv$$

$$= \int_3^6 \frac{1}{2} \frac{1}{v} u \Big|_{u=2}^{u=4} dv$$

$$= \int_3^6 \frac{1}{v} \, dv$$

$$= \ln|v| \Big|_3^6$$

$$= \ln 6 - \ln 3$$

$$= \ln\left(\frac{6}{3}\right)$$

$$= \ln 2$$

(11) Find the area of the first-quadrant region bounded by $y = x^3$, $y = 2x^3$, $x = y^3$, and $x = 4y^3$.

Rewrite the curves as $\frac{y}{x^3} = 1$, $\frac{y}{x^3} = 2$, $\frac{x}{y^3} = 1$, and $\frac{x}{y^3} = 4$.

Let $u = \frac{y}{x^3}$ and $v = \frac{x}{y^3}$.

$R: \ 1 \le u \le 2, \ 1 \le v \le 4$.

The integral is $A = \iint\limits_R 1 \, dx\, dy$

so the integrand $= 1$.

$$J_{(u,v) \to (x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} -3yx^{-4} & x^{-3} \\ y^{-3} & -3xy^{-4} \end{vmatrix}$$

$$= 9x^{-3}y^{-3} - x^{-3}y^{-3}$$

$$= 8x^{-3}y^{-3}$$

Continued $\longrightarrow$

(11) Cont'd We need to write $x^{-3}y^{-3}$ in terms of u and v.

$$u = \frac{y}{x^3} \quad \text{and} \quad v = \frac{x}{y^3} \Rightarrow uv = \frac{xy}{x^3y^3}$$

$$= x^{-2}y^{-2}$$

$$\Rightarrow (uv)^{3/2} = x^{-3}y^{-3}$$

So $J_{(u,v)\to(x,y)} = 8(uv)^{3/2}$

$$J_{(x,y)\to(u,v)} = \frac{1}{8}u^{-3/2}v^{-3/2}$$

$$|J_{(x,y)\to(u,v)}| = \frac{1}{8}u^{-3/2}v^{-3/2}$$

since $u>0$ and $v>0$.

$$A = \iint_R 1 \, dx\,dy$$

$$= \iint_R 1 \cdot |J_{(x,y)\to(u,v)}| \, du\,dv$$

$$= \int_1^4 \int_1^2 \frac{1}{8}u^{-3/2}v^{-3/2} \, du\,dv$$

$$= \frac{1}{8}\int_1^4 \Big[-2u^{-1/2}v^{-3/2}\Big]_{u=1}^{u=2} dv$$

Continued $\longrightarrow$

(11) Cont'd

$$= \frac{1}{8}\int_{1}^{4}\left[-\sqrt{2}\,r^{-3/2} + 2r^{-3/2}\right]dr$$

$$= \frac{1}{8}\int_{1}^{4}(2-\sqrt{2})\,r^{-3/2}\,dr$$

$$= \frac{1}{8}(2-\sqrt{2})\left[-2r^{-1/2}\right]_{1}^{4}$$

$$= \frac{1}{8}(2-\sqrt{2})\left[-1+2\right]$$

$$= \frac{2-\sqrt{2}}{8}$$

(13) Find the volume of the region bounded by the xy-plane, $z = x^2 + y^2$, and $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

In other words, evaluate $V = \iint_R (x^2+y^2)\,dx\,dy$ where R is the region inside $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Recall:
To parametrize the region inside $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
we use $x = ar\cos\theta$, $y = br\sin\theta$
where $0 \le r \le 1$, $0 \le \theta \le 2\pi$

Let $x = 3r\cos\theta$ $\quad y = 2r\sin\theta$

R: $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$$\text{Integrand} = x^2 + y^2$$
$$= 9r^2\cos^2\theta + 4r^2\sin^2\theta$$
$$= 5r^2\cos^2\theta + 4r^2\cos^2\theta + 4r^2\sin^2\theta$$
$$= 5r^2\,\frac{1+\cos 2\theta}{2} + 4r^2(\cos^2\theta + \sin^2\theta)$$
$$= \frac{5}{2}r^2(1+\cos 2\theta) + 4r^2$$

Continued $\longrightarrow$

13 Cont'd

$$J_{(x,y)\to(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} 3\cos\theta & -3r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{vmatrix}$$

$$= 6r\cos^2\theta + 6r\sin^2\theta$$

$$= 6r\,(\cos^2\theta + \sin^2\theta)$$

$$= 6r$$

$|J_{(x,y)\to(r,\theta)}| = 6r$ since $r \geq 0$.

$$V = \iint_R (x^2+y^2)\,dx\,dy$$

$$= \iint_R \left[\frac{5}{2}r^2(1+\cos 2\theta) + 4r^2\right]\underbrace{6r}_{|J_{(x,y)\to(r,\theta)}|}\,dr\,d\theta$$

$$= 6\int_0^{2\pi}\int_0^1 \left[\frac{5}{2}r^3(1+\cos 2\theta) + 4r^3\right]dr\,d\theta$$

$$= 6\int_0^{2\pi}\left[\frac{5}{8}r^4(1+\cos 2\theta) + r^4\right]_{r=0}^{r=1} d\theta$$

Continued →

(13) Cont'd

$$= 6 \int_0^{2\pi} \left[\frac{5}{8} (1 + \cos 2\theta) + 1 \right] d\theta$$

$$= 6 \left[\frac{5}{8} \left(\theta + \frac{\sin 2\theta}{2} \right) + \theta \right]_0^{2\pi}$$

$$= 6 \left[\frac{5}{8} (2\pi) + 2\pi \right]$$

$$= 6 \left[\frac{13}{8} (2\pi) \right]$$

$$= \frac{39\pi}{2}$$

(17) Evaluate $2\iint\limits_S e^{-3u^2-v^2}\,du\,dv$

where S is the region inside

$$3u^2+v^2=3.$$

S: region inside $u^2+\frac{v^2}{3}=1$

Recall: To parametrize the region inside $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we use

$$x = ar\cos\theta \,,\; y = br\sin\theta$$
$$0\le r\le 1,\quad 0\le\theta\le 2\pi$$

Let $u = r\cos\theta \quad v=\sqrt{3}\,r\sin\theta$

S: $0\le r\le 1 \,,\; 0\le\theta\le 2\pi$

$$\begin{aligned}-3u^2-v^2 &= -3r^2\cos^2\theta - 3r^2\sin^2\theta\\ &= -3r^2(\cos^2\theta+\sin^2\theta)\\ &= -3r^2\end{aligned}$$

$\Rightarrow$ integrand $= e^{-3r^2}$

Continued $\longrightarrow$

(17) Cont'd

$$J_{(u,v)\to(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sqrt{3}\sin\theta & \sqrt{3}r\cos\theta \end{vmatrix}$$

$$= \sqrt{3}r\cos^2\theta + \sqrt{3}r\sin^2\theta$$

$$= \sqrt{3}r(\cos^2\theta + \sin^2\theta)$$

$$= \sqrt{3}r$$

$|J_{(u,v)\to(r,\theta)}| = \sqrt{3}r$ since $r \geq 0$

$$2\iint_S e^{-3u^2-v^2}\,du\,dv$$

$$= 2\iint_S e^{-3r^2} |J_{(u,v)\to(r,\theta)}|\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^1 e^{-3r^2}\sqrt{3}r\,dr\,d\theta$$

$$\boxed{\begin{aligned} &\text{Sub } w = -3r^2 \\ &dw = -6r\,dr \\ &-\tfrac{1}{6}dw = r\,dr \\ &r=0 \Rightarrow w=0 \\ &r=1 \Rightarrow w=-3 \end{aligned}}$$

Continued

17 Cont'd

$$= -\frac{2\sqrt{3}}{6}\int_0^{2\pi}\int_0^{-3} e^w \, dw \, d\theta$$

$$= -\frac{2\sqrt{3}}{6}\int_0^{2\pi} e^w \Big|_{w=0}^{w=-3} \, d\theta$$

$$= -\frac{2\sqrt{3}}{6}\left[e^{-3}-1\right]\int_0^{2\pi} d\theta$$

$$= \frac{2\sqrt{3}}{6}\left(1-e^{-3}\right)\theta\Big|_0^{2\pi}$$

$$= \frac{2\sqrt{3}\left(1-e^{-3}\right)2\pi}{6}$$

$$= \frac{2\sqrt{3}\left(1-e^{-3}\right)\pi}{3}$$