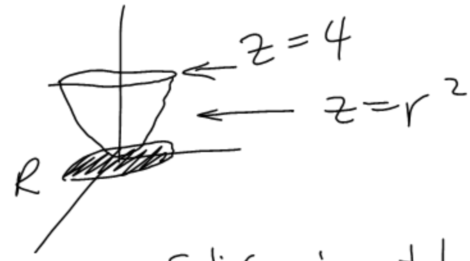
(1)



Slice in the z-direction:

$$r^2 \le z \le 4$$

Project on xy-plane:

$$z = z$$
$$r^2 = 4$$
$$r = \pm 2$$
$$r = 2$$

$$R: \quad 0 \le r \le 2$$
$$0 \le \theta \le 2\pi$$

$$V = \iiint_T dV$$

$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 rz \Big|_{z=r^2}^{z=4} dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$$

Continued →

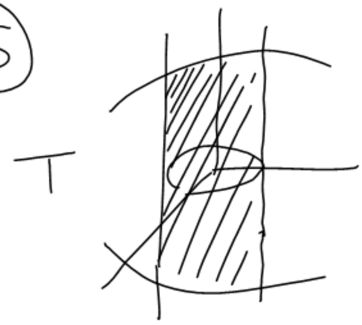
① Cont'd

$$= \int_0^{2\pi} 4\, d\theta$$

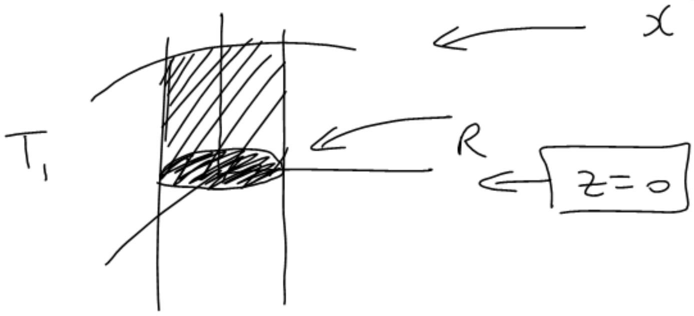
$$= 4\theta \Big|_0^{2\pi}$$

$$= 8\pi$$

(5)



By symmetry the volume is double the following volume:



$$x^2+y^2+z^2 = 4$$
$$r^2+z^2 = 4$$
$$z^2 = 4-r^2$$
$$z = \pm\sqrt{4-r^2}$$
$$z = \sqrt{4-r^2}$$

slice in the z-direction:
$$0 \le z \le \sqrt{4-r^2}$$
Project on xy-plane:
$$x^2+y^2 = 1$$
$$r^2 = 1$$
$$r = \pm 1$$
$$r = 1$$

R: $0 \le r \le 1$
$0 \le \theta \le 2\pi$

$$V = 2 \iiint_{T_1} dV$$
$$= 2 \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

Continued →

⑤ Cont'd

$$= 2\int_0^{2\pi}\int_0^1 rz\Big|_{z=0}^{z=\sqrt{4-r^2}}\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^1 r\sqrt{4-r^2}\,dr\,d\theta$$

Sub $u = 4-r^2$
$du = -2r\,dr$
$-\frac{du}{2} = r\,dr$
$r=0 \Rightarrow u=4$
$r=1 \Rightarrow u=3$

$$= -\int_0^{2\pi}\int_4^3 \sqrt{u}\,du\,d\theta$$

$$= -\int_0^{2\pi} \frac{2}{3}u^{3/2}\Big|_4^3\,d\theta$$

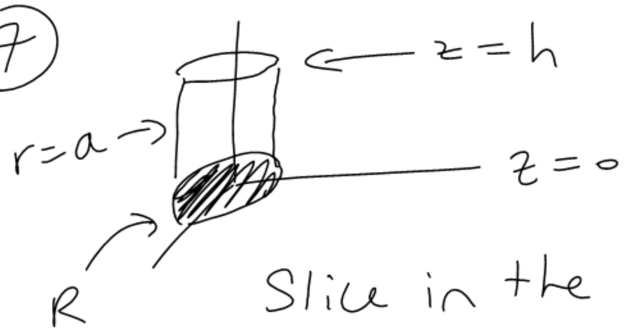
$$= -\int_0^{2\pi} \frac{2}{3}\left(3\sqrt{3}-8\right)d\theta$$

$$= -\frac{2}{3}\left(3\sqrt{3}-8\right)\theta\Big|_0^{2\pi}$$

$$= -\frac{4\pi\left(3\sqrt{3}-8\right)}{3}$$

$$= \frac{4\pi\left(8-3\sqrt{3}\right)}{3}$$

(7)



Slice in the z-direction:

$$0 \le z \le h$$

Project on xy-plane:

$$r = a$$

$$R: \quad 0 \le r \le a, \quad 0 \le \theta \le 2\pi$$

$$m = \iiint_T \delta \, dV \qquad \boxed{\text{Given } \delta = z}$$

$$= \int_0^{2\pi} \int_0^a \int_0^h z \, r \, dz \, dr \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^a z^2 r \Big|_{z=0}^{z=h} dr \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^a h^2 r \, dr \, d\theta$$

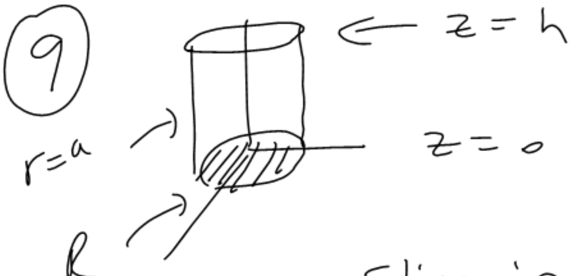
$$= \frac{1}{4} \int_0^{2\pi} h^2 r^2 \Big|_{r=0}^{r=a} d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} h^2 a^2 \, d\theta$$

$$= \frac{h^2 a^2}{4} \theta \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{a^2 h^2 \pi}{2}$$

9



Slice in the z-direction:

$$0 \le z \le h$$

Project on xy-plane:

$$r = a$$

$$R: \quad 0 \le r \le a, \quad 0 \le \theta \le 2\pi$$

$$I_z = \iiint_T \underbrace{(x^2+y^2)}_{r^2} \, \delta \, dV$$

$dV \leftarrow r\,dz\,dr\,d\theta$

$\delta \leftarrow z$ (given)

$$= \int_0^{2\pi}\int_0^a\int_0^h r^3 z \, dz\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^a r^3 z^2 \Big|_{z=0}^{z=h} dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^a r^3 h^2 \, dr\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} r^4 h^2 \Big|_{r=0}^{r=a} d\theta$$

$$= \frac{1}{8} a^4 h^2 \, \theta \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{a^4 h^2 \pi}{4}$$

(11)

$z = 9 - x^2 - y^2$

$z = 9 - r^2$

$z = 0$

Slice in the z-direction:

$$0 \le z \le 9 - r^2$$

Project on xy-plane:

$$z = z$$
$$9 - r^2 = 0$$
$$9 = r^2$$
$$r = \pm 3$$
$$r = 3$$

$$R: \quad \begin{array}{l} 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{array}$$

$$V = \iiint_T dV$$

$$= \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^3 rz \Big|_{z=0}^{z=9-r^2} dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^3 \underbrace{r(9-r^2)}_{9r - r^3} \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_{r=0}^{r=3} d\theta$$

Continued →

(11) Cont'd

$$= \int_0^{2\pi} \frac{81}{4} \, d\theta$$

$$= \frac{81}{4} \theta \Big|_0^{2\pi}$$

$$= \frac{81\pi}{2}$$

Because δ is constant and the solid region T is symmetric about $x=0$, we conclude $\bar{x}=0$.
Similarly $\bar{y}=0$.

$$\delta = 1 \Rightarrow m = \iiint_T \delta \, dV$$

$$= \iiint_T dV$$

$$= \frac{81\pi}{2}$$

$$\bar{z} = \frac{1}{m} \iiint_T z \overset{1}{\delta} \, dV \longleftarrow r\,dz\,dr\,d\theta$$

$$= \frac{2}{81\pi} \int_0^{2\pi} \int_0^{3} \int_0^{9-r^2} z r \, dz\, dr\, d\theta$$

$$= \frac{1}{81\pi} \int_0^{2\pi} \int_0^{3} z^2 r \Big|_{z=0}^{z=9-r^2} dr\, d\theta$$

$$= \frac{1}{81\pi} \int_0^{2\pi} \int_0^{3} (9-r^2)^2 r \, dr\, d\theta$$

Continued →

(11) Cont'd

$$= \frac{1}{81\pi} \int_0^{2\pi} \int_0^3 \underbrace{(81 - 18r^2 + r^4)\, r}_{81r - 18r^3 + r^5} \, dr \, d\theta$$

$$= \frac{1}{81\pi} \int_0^{2\pi} \left[\frac{81r^2}{2} - \frac{18r^4}{4} + \frac{r^6}{6} \right]_0^3 d\theta$$

$$= \frac{1}{81\pi} \int_0^{2\pi} \frac{243}{2} \, d\theta$$

$$= \frac{3}{2\pi} \, \theta \Big|_0^{2\pi}$$

$$= 3$$

The centroid is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 3)$.

(13)

$\leftarrow z = 12 - x^2 - 2y^2$

$\leftarrow z = 2x^2 + y^2$

Slice in the z-direction:

$$2x^2 + y^2 \le z \le 12 - x^2 - 2y^2$$

(will be converted to cylindrical coordinates later)

Project on xy-plane:

$$z = z$$
$$2x^2 + y^2 = 12 - x^2 - 2y^2$$
$$3x^2 + 3y^2 = 12$$
$$3(x^2+y^2) = 12$$
$$r^2 = 4$$
$$r = \pm 2$$
$$r = 2$$

$R:$
$$0 \le r \le 2$$
$$0 \le \theta \le 2\pi$$

$$V = \iiint_T dV$$

$$= \int_0^{2\pi} \int_0^{2} \int_{2x^2+y^2}^{12-x^2-2y^2} r \, dz \, dr \, d\theta$$

Continued $\longrightarrow$

(13) Cont'd

$$= \int_0^{2\pi}\int_0^2 rz \Big|_{z=2x^2+y^2}^{z=12-x^2-2y^2} \, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r\left(12-x^2-2y^2-2x^2-y^2\right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \underbrace{r\left(12-3x^2-3y^2\right)}_{\substack{= r(12-3r^2) \\ = 12r-3r^3}} \, dr\,d\theta$$

$$= \int_0^{2\pi} \left[6r^2 - \frac{3r^4}{4}\right]_0^2 d\theta$$

$$= \int_0^{2\pi} 12 \, d\theta$$

$$= 12\theta \Big|_0^{2\pi}$$

$$= 24\pi$$

(15)

$z = x^2+y^2$

$z = r^2$

$x^2+y^2+z^2 = 2$

$z^2 = 2-r^2$

$z = \pm\sqrt{2-r^2}$

$z = \sqrt{2-r^2}$

Slice in the z-direction:

$$r^2 \le z \le \sqrt{2-r^2}$$

Project on xy-plane:

$$z = z$$

$$r^2 = \sqrt{2-r^2}$$

Square both sides: $r^4 = 2-r^2$

$$r^4+r^2-2 = 0$$

$$(r^2+2)(r^2-1) = 0$$

$r^2+2=0$ no solution

$r^2-1=0$

$r^2=1$

$r=\pm 1$

$r=1$

R: $0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$V = \iiint_T dV$$

$$= \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

Continued →

(15) Cont'd

$$= \int_0^{2\pi}\int_0^1 rz \Big|_{z=r^2}^{z=\sqrt{2-r^2}} \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(r\sqrt{2-r^2} - r^3\right) dr\, d\theta$$

$$\boxed{\begin{aligned} &\text{Let } I = \int r\sqrt{2-r^2}\, dr \\ &\quad \text{Sub } u = 2-r^2 \\ &\quad du = -2r\,dr \\ &\quad -\frac{du}{2} = r\,dr \\ &I = -\frac{1}{2}\int \sqrt{u}\, du \\ &\quad = -\frac{1}{2}\,\frac{2}{3}\, u^{3/2} + C \\ &\quad = -\frac{1}{3}(2-r^2)^{3/2} + C \end{aligned}}$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3} - \frac{1}{4} + \frac{2^{3/2}}{3}\right] d\theta$$

$$= \left[\frac{-4}{12} - \frac{3}{12} + \frac{8\sqrt{2}}{12}\right] \theta \Big|_0^{2\pi}$$

$$= \frac{8\sqrt{2}-7}{12}(2\pi)$$

$$= \frac{(8\sqrt{2}-7)\pi}{6}$$

(21)

$$x^2+y^2+z^2 = a^2$$
$$\rho^2 = a^2$$
$$\rho = a$$

$$\boxed{T:\ 0 \le \rho \le a,\ 0 \le \phi \le \tfrac{\pi}{2},\ 0 \le \theta \le 2\pi}$$

Homogeneous means δ is constant.

$\boxed{\text{Let } \delta = K}$

Because δ is constant and the solid region T is symmetric about $x=0$, we conclude $\bar{x} = 0$.
Similarly $\bar{y} = 0$.

$$V_{sphere} = \frac{4}{3}\pi r^3$$
$$V_{hemisphere} = \frac{2}{3}\pi r^3 \quad \text{by symmetry}$$
$$= \frac{2}{3}\pi a^3$$

$$\delta = K \implies m_{hemisphere} = \iiint_T \delta\, dV$$
$$= k\iiint_T dV$$
$$= \frac{2}{3}\pi a^3 k$$

$$\bar{z} = \frac{1}{m}\iiint_T z\,\delta\, dV$$

$\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

K

$z = \rho\cos\phi$

Continued →

(21) Cont'd

$$\bar{z} = \frac{3\cancel{k}}{2\pi a^3 \cancel{k}} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{a} \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{3}{8\pi a^3} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \rho^4 \cos\phi \sin\phi \Big|_{\rho=0}^{\rho=a} d\phi \, d\theta$$

$$= \frac{3}{8\pi a^3} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} a^4 \cos\phi \sin\phi \, d\phi \, d\theta$$

Recall	$\sin 2\phi = 2\sin\phi\cos\phi$
	$\frac{\sin 2\phi}{2} = \sin\phi\cos\phi$

$$= \frac{3}{16\pi a^3} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} a^4 \sin 2\phi \, d\phi \, d\theta$$

$$= \frac{3}{16\pi a^3} \int_0^{2\pi} -\frac{a^4 \cos 2\phi}{2} \Big|_{\phi=0}^{\phi=\frac{\pi}{2}} d\theta$$

$$= \frac{3}{16\pi a^3} \int_0^{2\pi} \left(\frac{a^4}{2} + \frac{a^4}{2} \right) d\theta$$

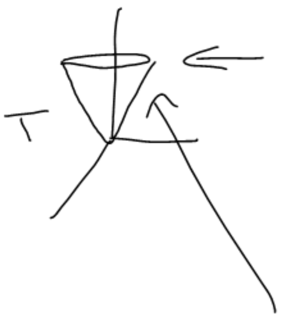
$$= \frac{3}{16\pi a^3} (a^4) \, \theta \Big|_0^{2\pi}$$

$$= \frac{3a}{16\pi} (2\pi)$$

$$= \frac{3a}{8}$$

The Centroid is $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{3a}{8}\right)$.

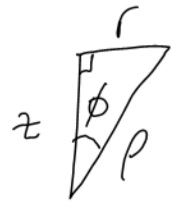
(23)



$z = 1$

$\rho\cos\phi = 1$

$\rho = \sec\phi$



$z = r$

$\rho\cos\phi = \rho\sin\phi$

$\cos\phi = \sin\phi$

$1 = \tan\phi$

$\arctan 1 = \phi$

$\phi = \frac{\pi}{4}$

T: $0 \le \rho \le \sec\phi$

$0 \le \phi \le \frac{\pi}{4}$

$0 \le \theta \le 2\pi$

$$V = \iiint_T dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^3 \sin\phi}{3}\Bigg|_{\rho=0}^{\rho=\sec\phi} d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\sec^3\phi \sin\phi}{3} \, d\phi \, d\theta$$

Continued →

(23) Cont'd

$$\boxed{\begin{aligned} &\sec\phi\sin\phi = \frac{\sin\phi}{\cos\phi} = \tan\phi \\ &\sec^3\phi\sin\phi = \tan\phi\sec^2\phi \end{aligned}}$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} \tan\phi\sec^2\phi \, d\phi \, d\theta$$

$$\boxed{\begin{aligned} &\text{Sub } u = \tan\phi \\ &\quad du = \sec^2\phi \, d\phi \\ &\phi = 0 \Rightarrow u = 0 \\ &\phi = \frac{\pi}{4} \Rightarrow u = 1 \end{aligned}}$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{1} u \, du \, d\theta$$

$$= \frac{1}{6}\int_0^{2\pi} u^2 \Big|_{u=0}^{u=1} d\theta$$

$$= \frac{1}{6}\int_0^{2\pi} d\theta$$

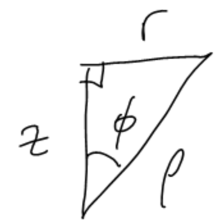
$$= \frac{1}{6}\theta \Big|_0^{2\pi}$$

$$= \frac{\pi}{3}$$

(25)

$\leftarrow \rho = a$

$\leftarrow z = r$



$$\rho\cos\phi = \rho\sin\phi$$
$$\cos\phi = \sin\phi$$
$$1 = \tan\phi$$
$$\arctan 1 = \phi$$
$$\phi = \frac{\pi}{4}$$

$$T: \quad 0 \le \rho \le a$$
$$0 \le \phi \le \frac{\pi}{4}$$
$$0 \le \theta \le 2\pi$$

$$V = \iiint_T dV$$
$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} \rho^3 \sin\phi \Big|_{\rho=0}^{\rho=a} d\phi \, d\theta$$
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} a^3 \sin\phi \, d\phi \, d\theta$$
$$= \frac{1}{3}\int_0^{2\pi} \Big[-a^3\cos\phi\Big]_{\phi=0}^{\phi=\pi/4} d\theta$$

Continued $\longrightarrow$

(25) Cont'd $= \frac{a^3}{3} \int_0^{2\pi} \left[-\frac{\sqrt{2}}{2} + 1 \right] d\theta$

$= \frac{a^3}{3} \left(\frac{2-\sqrt{2}}{2} \right) \theta \Big|_0^{2\pi}$

$= \frac{a^3 (2-\sqrt{2}) \pi}{3}$

Uniform means δ is constant.

$\boxed{\text{Let } \delta = k.}$

Because δ is constant and the solid region T is symmetric about $x=0$, we conclude $\bar{x} = 0$.

Similarly $\bar{y} = 0$.

$$m = \iiint_T \delta \, dV$$

$$= k \iiint_T dV$$

$$= kV$$

$$\bar{z} = \frac{1}{m} \iiint_T z \, \delta \, dV$$

$\rho \cos\phi$ k $\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$= \frac{1}{V} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{a} \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

Continued $\longrightarrow$

(25) Cont'd

$$= \frac{1}{4V} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \rho^4 \cos\phi \sin\phi \Big|_{\rho=0}^{\rho=a} d\phi\, d\theta$$

$$= \frac{a^4}{4V} \int_0^{2\pi} \int_0^{\pi/4} \cos\phi \sin\phi \, d\phi\, d\theta$$

$$= \frac{a^4}{4V} \int_0^{2\pi} \int_0^{\pi/4} \frac{\sin 2\phi}{2} \, d\phi\, d\theta$$

$$= \frac{a^4}{4V} \int_0^{2\pi} \left[-\frac{\cos 2\phi}{4} \right]_{\phi=0}^{\phi=\frac{\pi}{4}} d\theta$$

$$= \frac{a^4}{4V} \int_0^{2\pi} \frac{1}{4} \, d\theta$$

$$= \frac{a^4}{16V} \theta \Big|_0^{2\pi}$$

$$= \frac{a^4 \pi}{8V}$$

$$= \frac{a^4 \pi}{8} \cdot \frac{3}{a^3 (2-\sqrt{2}) \pi}$$

$$= \frac{3a}{8(2-\sqrt{2})} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}}$$

$$= \frac{3a(2+\sqrt{2})}{16}$$ Centroid is $\left(0, 0, \frac{3a(2+\sqrt{2})}{16}\right)$.