7

$R: \quad 0 \le y \le x^2$
$\quad\;\; 0 \le x \le 2$

$$m = \iint_R \delta \, dA \qquad \boxed{\delta = 1}$$

$$= \int_0^2 \int_0^{x^2} dy \, dx$$

$$= \int_0^2 y \Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^2 x^2 dx$$

$$= \frac{x^3}{3}\Big|_0^2$$

$$= \frac{8}{3}$$

$$\bar{x} = \frac{1}{m} \iint_R x \delta \, dA$$

$$= \frac{3}{8} \int_0^2 \int_0^{x^2} x \, dy \, dx$$

$$= \frac{3}{8} \int_0^2 xy \Big|_{y=0}^{y=x^2} dx$$

Continued →

7 Cont'd

$$= \frac{3}{8}\int_0^2 x^3\,dx$$

$$= \frac{3}{8}\ \frac{x^4}{4}\Big|_0^2$$

$$= \frac{3}{2}$$

$$\bar{y} = \frac{1}{m}\iint_R y\,\delta\,dA$$

$$= \frac{3}{8}\int_0^2\int_0^{x^2} y\,dy\,dx$$

$$= \frac{3}{8}\int_0^2 \frac{y^2}{2}\Big|_{y=0}^{y=x^2}\,dx$$

$$= \frac{3}{8}\int_0^2 \frac{x^4}{2}\,dx$$

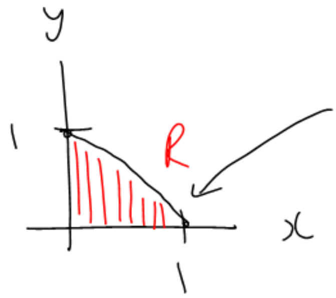
$$= \frac{3}{8}\ \frac{x^5}{10}\Big|_0^2$$

$$= \frac{3}{8}\left(\frac{32}{10}\right)$$

$$= \frac{6}{5}$$

The centroid is $(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{6}{5}\right)$.

(11)



$R: \quad 0 \le y \le 1-x$
$\qquad 0 \le x \le 1$

Because R is symmetric about the line $y=x$ and $\delta(y,x) = yx = xy = \delta(x,y)$, we can conclude that $\bar{y} = \bar{x}$.

$$m = \iint_R \delta \, dA$$
$$= \int_0^1 \int_0^{1-x} xy \, dy \, dx$$
$$= \int_0^1 \frac{xy^2}{2} \Bigg|_{y=0}^{y=1-x} dx$$
$$= \frac{1}{2} \int_0^1 x(1-x)^2 \, dx$$
$$= \frac{1}{2} \int_0^1 x(1-2x+x^2) \, dx$$
$$= \frac{1}{2} \int_0^1 (x - 2x^2 + x^3) \, dx$$
$$= \frac{1}{2} \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1$$
$$= \frac{1}{2} \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right]$$
$$= \frac{1}{24}$$

Continued $\longrightarrow$

(11) Cont'd

$$\bar{x} = \frac{1}{m} \iint_R x\, \delta\, dA$$

$$= 24 \int_0^1 \int_0^{1-x} x^2 y \; dy\, dx$$

$$= 24 \int_0^1 \frac{x^2 y^2}{2} \Big|_{y=0}^{y=1-x} dx$$

$$= 12 \int_0^1 x^2 (1-x)^2 \; dx$$

$$= 12 \int_0^1 x^2 (1 - 2x + x^2)\, dx$$

$$= 12 \int_0^1 (x^2 - 2x^3 + x^4)\, dx$$

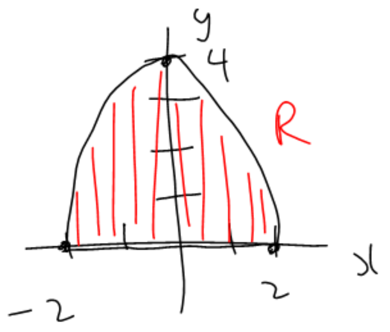
$$= 12 \left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \right]_0^1$$

$$= 12 \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right]$$

$$= \frac{2}{5}$$

The centroid is $(\bar{x}, \bar{y}) = \left(\frac{2}{5}, \frac{2}{5}\right)$.

(13)



$R: \quad 0 \le y \le 4 - x^2$

$\quad -2 \le x \le 2$

Because R is symmetric about the line $x=0$ and $\delta(-x,y) = y = \delta(x,y)$, we can conclude that $\bar{x} = 0$.

$$m = \iint_R \delta \, dA$$

$$= \int_{-2}^{2} \int_{0}^{4-x^2} y \, dy \, dx$$

$$= \int_{-2}^{2} \left. \frac{y^2}{2} \right|_{y=0}^{y=4-x^2} dx$$

$$= \frac{1}{2} \int_{-2}^{2} (4-x^2)^2 \, dx$$

$$= \frac{1}{2} \int_{-2}^{2} (16 - 8x^2 + x^4) \, dx$$

$$= \frac{1}{2} \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-2}^{2}$$

$$= \frac{1}{2} \left[32 - \frac{64}{3} + \frac{32}{5} - \left(-32 + \frac{64}{3} - \frac{32}{5} \right) \right]$$

$$= \frac{256}{15}$$

Continued →

(13) Cont'd

$$\bar{y} = \frac{1}{m} \iint_R y \, \delta \, dA$$

$$= \frac{15}{256} \int_{-2}^{2} \int_{0}^{4-x^2} y^2 \, dy \, dx$$

$$= \frac{15}{256} \int_{-2}^{2} \frac{y^3}{3} \Big|_{y=0}^{y=4-x^2} dx$$

$$= \frac{5}{256} \int_{-2}^{2} (4-x^2)^3 \, dx$$

$$\boxed{\begin{aligned} (4-x^2)^3 &= (4-x^2)(4-x^2)^2 \\ &= (4-x^2)(16-8x^2+x^4) \\ &= 64-32x^2+4x^4 \\ &\quad -16x^2+8x^4-x^6 \\ &= 64-48x^2+12x^4-x^6 \end{aligned}}$$

$$= \frac{5}{256} \int_{-2}^{2} (64-48x^2+12x^4-x^6) \, dx$$

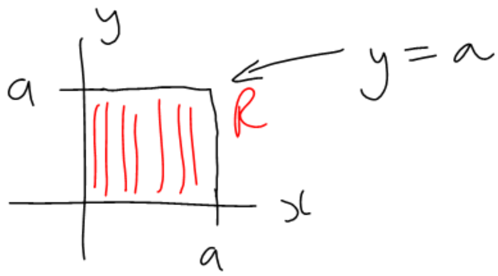
$$= \frac{5}{256} \left[64x - 16x^3 + \frac{12x^5}{5} - \frac{x^7}{7} \right]_{-2}^{2}$$

$$= \frac{5}{256} \left[128 - 128 + \frac{384}{5} - \frac{128}{7} - \left(-128 + 128 - \frac{384}{5} + \frac{128}{7} \right) \right]$$

$$= \frac{16}{7}$$

The centroid is $(\bar{x}, \bar{y}) = (0, \frac{16}{7})$.

(21)



$R: \; 0 \le y \le a$
$\quad\; 0 \le x \le a$

Because R is symmetric about the line $y = x$ and $\delta(y, x) = y + x = x + y = \delta(x, y)$, we can conclude that $\bar{y} = \bar{x}$.

$$m = \iint_R \delta \, dA$$

$$= \int_0^a \int_0^a (x+y) \, dy \, dx$$

$$= \int_0^a \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=a} dx$$

$$= \int_0^a \left[ax + \frac{a^2}{2} \right] dx$$

$$= \left[\frac{ax^2}{2} + \frac{a^2}{2} x \right]_{x=0}^{x=a}$$

$$= \frac{a^3}{2} + \frac{a^3}{2}$$

$$= a^3$$

$$\bar{x} = \frac{1}{m} \iint_R x \, \delta \, dA$$

$$= \frac{1}{a^3} \int_0^a \int_0^a x(x+y) \, dy \, dx$$

Continued $\rightarrow$

(21) Cont'd

$$= \frac{1}{a^3} \int_0^a \int_0^a (x^2 + xy)\, dy\, dx$$

$$= \frac{1}{a^3} \int_0^a \left[x^2 y + \frac{xy^2}{2} \right]_{y=0}^{y=a} dx$$

$$= \frac{1}{a^3} \int_0^a \left[ax^2 + \frac{a^2}{2} x \right] dx$$

$$= \frac{1}{a^3} \left[\frac{ax^3}{3} + \frac{a^2 x^2}{4} \right]_{x=0}^{x=a}$$

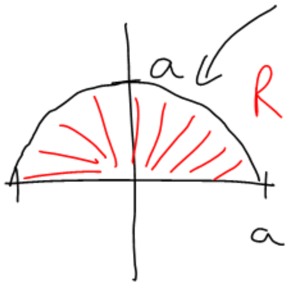
$$= \frac{1}{a^3} \left[\frac{a^4}{3} + \frac{a^4}{4} \right]$$

$$= \frac{1}{a^3} \left(\frac{7a^4}{12} \right)$$

$$= \frac{7a}{12}$$

The centroid is $(\bar{x}, \bar{y}) = \left(\frac{7a}{12}, \frac{7a}{12} \right)$.

(27)



$x^2+y^2=a^2$
$r^2=a^2$
$r=\pm a$
$r=a$

Polar coordinates

$R: \ 0 \le r \le a$
$\quad\ \ 0 \le \theta \le \pi$

$$m = \iint_R \delta \, dA \qquad \boxed{\delta = r, \quad dA = r\,dr\,d\theta}$$

$$= \int_0^{\pi}\int_0^{a} r^2 \, dr\, d\theta$$

$$= \int_0^{\pi} \frac{r^3}{3}\Big|_{r=0}^{r=a} d\theta$$

$$= \int_0^{\pi} \frac{a^3}{3} \, d\theta$$

$$= \frac{a^3}{3}\theta \Big|_0^{\pi}$$

$$= \frac{a^3\pi}{3}$$

Recall $r = \sqrt{x^2+y^2}$.
Because R is symmetric about $x=0$
and $\delta(-x,y) = \sqrt{x^2+y^2} = \delta(x,y)$, we
can conclude that $\bar{x} = 0$.

Continued $\longrightarrow$

(27) Cont'd

$$\bar{y} = \frac{1}{m} \iint_R y \, \delta \, dA$$

$$= \frac{3}{a^3 \pi} \int_0^{\pi} \int_0^{a} r\sin\theta \cdot r \cdot r \, dr \, d\theta$$

$$= \frac{3}{a^3 \pi} \int_0^{\pi} \int_0^{a} r^3 \sin\theta \, dr \, d\theta$$

$$= \frac{3}{a^3 \pi} \int_0^{\pi} \frac{r^4 \sin\theta}{4} \Bigg|_{r=0}^{r=a} d\theta$$

$$= \frac{3}{a^3 \pi} \int_0^{\pi} \frac{a^4 \sin\theta}{4} \, d\theta$$

$$= \frac{3a}{4\pi} \Big[-\cos\theta \Big]_0^{\pi}$$

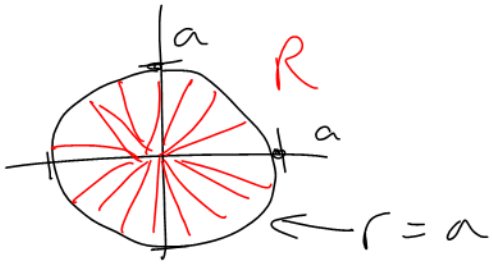
$$= \frac{3a}{4\pi} \left[1 - (-1) \right]$$

$$= \frac{3a}{4\pi} (2)$$

$$= \frac{3a}{2\pi}$$

The centroid is $(\bar{x}, \bar{y}) = \left(0, \frac{3a}{2\pi}\right)$.

(31)



R: $0 \le r \le a$

$0 \le \theta \le 2\pi$

$$I_0 = \iint_R \underbrace{(x^2+y^2)}_{r^2} \underbrace{\delta}_{r^n} \underbrace{dA}_{r\,dr\,d\theta}$$

$$= \int_0^{2\pi} \int_0^a r^{n+3} \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{r^{n+4}}{n+4} \Bigg|_{r=0}^{r=a} d\theta$$

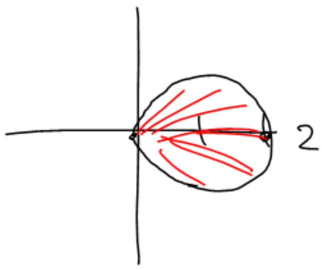
$$= \int_0^{2\pi} \frac{a^{n+4}}{n+4} \, d\theta$$

$$= \frac{a^{n+4}}{n+4} \theta \Bigg|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{a^{n+4}(2\pi)}{n+4}$$

$$= \frac{2a^{n+4}\pi}{n+4}$$

(33)



θ	$r = 2\cos\theta$
$-\frac{\pi}{2}$	0
0	2
$\frac{\pi}{2}$	0

The curve is traced out over $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

R: $0 \le r \le 2\cos\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$I_0 = \iint_R \underbrace{(x^2+y^2)}_{r^2} \underset{k}{\delta}\, \underset{r\,dr\,d\theta}{dA}$$

$$= k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^3\, dr\, d\theta$$

$$= k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\Bigg|_{r=0}^{r=2\cos\theta} d\theta$$

$$= k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^4\theta\, d\theta$$

Continued →

(33) Cont'd

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$2\cos^2\theta = 1+\cos 2\theta$$

$$4\cos^4\theta = (2\cos^2\theta)^2$$

$$= (1+\cos 2\theta)^2$$

$$= 1 + 2\cos 2\theta + \cos^2 2\theta$$

$$= 1+2\cos 2\theta + \frac{1+\cos 4\theta}{2}$$

$$= \frac{3}{2} + 2\cos 2\theta + \frac{\cos 4\theta}{2}$$

$$= k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos 2\theta + \frac{\cos 4\theta}{2}\right) d\theta$$

$$= k\left[\frac{3}{2}\theta + \sin 2\theta + \frac{\sin 4\theta}{8}\right]_{-\pi/2}^{\pi/2}$$

$$= k\left[\frac{3\pi}{4} - \left(-\frac{3\pi}{4}\right)\right]$$

$$= \frac{3\pi k}{2}$$