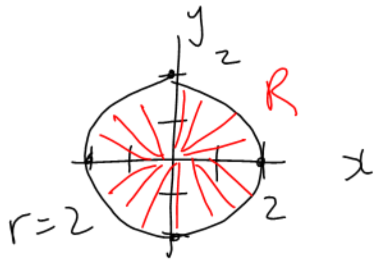
(9)



$R: \quad 0 \le r \le 2$
$\quad\;\; 0 \le \theta \le 2\pi$

$$z = \sqrt{x^2+y^2}$$
$$= r$$

$$dA = r\,dr\,d\theta$$

$$V = \iint_R z\,dA$$

$$= \int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta$$

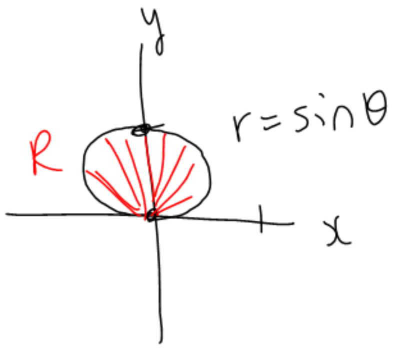
$$= \int_0^{2\pi} \frac{r^3}{3}\Bigg|_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} \frac{8}{3}\,d\theta$$

$$= \frac{8}{3}\theta\Bigg|_0^{2\pi}$$

$$= \frac{16\pi}{3}$$

(11)



To plot $r=\sin\theta$:

θ	r
0	0
$\frac{\pi}{2}$	1
π	0

Note: $r=\sin\theta$ is traced out over the interval $0\le\theta\le\pi$.

$$R: \quad \begin{array}{l} 0\le r\le \sin\theta \\ 0\le\theta\le\pi \end{array}$$

$$z = 10+2x+3y$$
$$= 10+2r\cos\theta+3r\sin\theta$$

$$dA = r\,dr\,d\theta$$

$$V = \iint_R z\,dA$$
$$= \int_0^{\pi}\int_0^{\sin\theta}(10+2r\cos\theta+3r\sin\theta)\,r\,dr\,d\theta$$
$$= \int_0^{\pi}\int_0^{\sin\theta}(10r+2r^2\cos\theta+3r^2\sin\theta)\,dr\,d\theta$$
$$= \int_0^{\pi}\left[5r^2+\frac{2r^3}{3}\cos\theta+r^3\sin\theta\right]_{r=0}^{r=\sin\theta}d\theta$$
$$= \int_0^{\pi}\left(5\sin^2\theta+\frac{2}{3}\sin^3\theta\cos\theta+\sin^4\theta\right)d\theta$$

Continued $\rightarrow$

(11) Cont'd

$$= \underbrace{\int_0^{\pi} 5\sin^2\theta \, d\theta}_{I_1} + \underbrace{\int_0^{\pi} \frac{2}{3}\sin^3\theta\cos\theta \, d\theta}_{I_2} + \underbrace{\int_0^{\pi} \sin^4\theta \, d\theta}_{I_3}$$

$$I_1 = \int_0^{\pi} 5\sin^2\theta \, d\theta$$

$$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$$

$$= \frac{5}{2}\int_0^{\pi} (1-\cos 2\theta)\, d\theta$$

$$= \frac{5}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{5}{2}\left[\pi\right]$$

$$= \frac{5\pi}{2}$$

$$I_2 = \int_0^{\pi} \frac{2}{3}\sin^3\theta\cos\theta \, d\theta$$

$$\boxed{\begin{aligned} u &= \sin\theta \\ du &= \cos\theta\, d\theta \\ \theta = 0 &\Rightarrow u = 0 \\ \theta = \pi &\Rightarrow u = 0 \end{aligned}}$$

$$= \frac{2}{3}\int_0^{0} u^3 \, du$$

$$= \frac{2}{3} \left. \frac{u^4}{4} \right|_0^0$$

$$= 0$$

$$I_3 = \int_0^{\pi} \sin^4\theta \, d\theta$$

$$\boxed{\begin{aligned} \sin^2\theta &= \tfrac{1}{2}(1-\cos 2\theta) \\ \sin^4\theta &= (\sin^2\theta)^2 \\ &= \tfrac{1}{4}(1 - 2\cos 2\theta + \cos^2 2\theta) \\ &= \tfrac{1}{4}\left(1 - 2\cos 2\theta + \tfrac{1}{2}(1+\cos 4\theta)\right) \\ &= \frac{1}{4} - \frac{\cos 2\theta}{2} + \frac{1}{8} + \frac{\cos 4\theta}{8} \\ &= \frac{3}{8} - \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{8} \end{aligned}}$$

Continued →

(11) Cont'd

$$I_3 = \int_0^{\pi} \left[\frac{3}{8} - \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{8} \right] d\theta$$

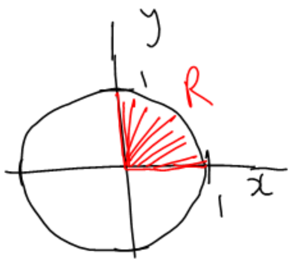
$$= \left[\frac{3}{8}\theta - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32} \right]_0^{\pi}$$

$$= \frac{3\pi}{8}$$

Therefore $V = \frac{5\pi}{2} + 0 + \frac{3\pi}{8}$

$$= \frac{23\pi}{8}$$

(13) R: $0 \le x \le \sqrt{1-y^2}$
$0 \le y \le 1$



$x = \sqrt{1-y^2}$
$x^2 = 1-y^2$
$x^2+y^2 = 1$

R: $0 \le r \le 1$
$0 \le \theta \le \frac{\pi}{2}$

$$f = \frac{1}{1+x^2+y^2} = \frac{1}{1+r^2}$$

$$dx\,dy = r\,dr\,d\theta$$

$$\text{Integral} = \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{1+r^2} r\,dr\,d\theta$$

Sub $u = 1+r^2$
$du = 2r\,dr$
$\frac{du}{2} = r\,dr$

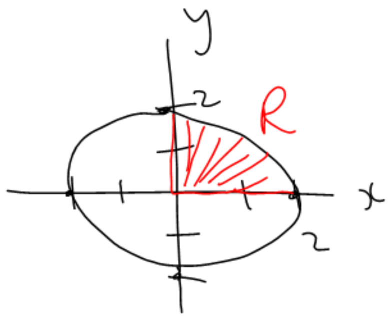
$r=0 \Rightarrow u=1$
$r=1 \Rightarrow u=2$

$$\int_0^1 \frac{1}{1+r^2} r\,dr = \frac{1}{2}\int_1^2 \frac{du}{u}$$
$$= \frac{1}{2}\ln|u| \Big|_{u=1}^{u=2}$$
$$= \frac{1}{2}\ln 2$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\ln 2 \, d\theta$$
$$= \frac{1}{2}\ln 2 \ \theta \Big|_0^{\pi/2}$$
$$= \frac{\pi \ln 2}{4}$$

(15) $R: \quad 0 \le y \le \sqrt{4-x^2}$

$0 \le x \le 2$



$$y = \sqrt{4-x^2}$$
$$y^2 = 4 - x^2$$
$$x^2 + y^2 = 4$$

$R: \quad 0 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$

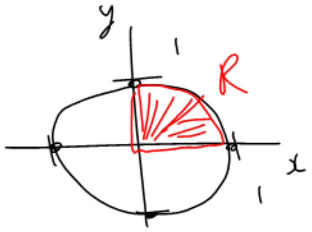
$$f = (x^2+y^2)^{3/2}$$
$$= (r^2)^{3/2}$$
$$= r^3$$

$$dy\,dx = r\,dr\,d\theta$$

$$\text{Integral} = \int_0^{\frac{\pi}{2}}\int_0^2 r^4\,dr\,d\theta$$
$$= \int_0^{\frac{\pi}{2}} \frac{r^5}{5}\Big|_0^2\,d\theta$$
$$= \int_0^{\frac{\pi}{2}} \frac{32}{5}\,d\theta$$
$$= \frac{32}{5}\theta\Big|_0^{\pi/2}$$
$$= \frac{16\pi}{5}$$

(17) $R: \quad 0 \le x \le \sqrt{1-y^2}$

$\quad\quad 0 \le y \le 1$



$x = \sqrt{1-y^2}$

$x^2 = 1-y^2$

$x^2+y^2 = 1$

$R: \quad 0 \le r \le 1$

$\quad\quad 0 \le \theta \le \frac{\pi}{2}$

$f = \sin(x^2+y^2)$

$\quad = \sin r^2$

$dx\,dy = r\,dr\,d\theta$

$$\text{Integral} = \int_0^{\pi/2}\int_0^1 r\sin r^2\, dr\, d\theta$$

Sub $u = r^2$

$du = 2r\,dr$

$\frac{du}{2} = r\,dr$

$r=0 \Rightarrow u=0$

$r=1 \Rightarrow u=1$

$$\int_0^1 r\sin r^2\, dr = \frac{1}{2}\int_0^1 \sin u\, du$$

$$= -\frac{1}{2}\cos u\Big|_0^1$$

$$= -\frac{1}{2}\cos 1 + \frac{1}{2}$$

$$= \frac{1}{2}(1-\cos 1)$$

$$= \int_0^{\pi/2} \frac{1}{2}(1-\cos 1)\, d\theta$$

Continued →

17 Cont'd

$$= \frac{1}{2}(1-\cos 1)\theta \Big|_0^{\pi/2}$$

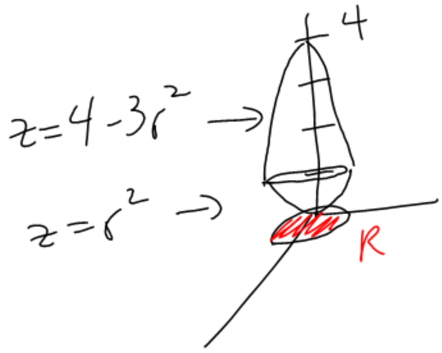
$$= \frac{1}{2}(1-\cos 1)\frac{\pi}{2}$$

$$= \frac{\pi(1-\cos 1)}{4}$$

$$\approx 0.36$$

(27) Intersection:

$$z = z$$
$$x^2+y^2 = 4-3x^2-3y^2$$
$$r^2 = 4-3r^2$$
$$4r^2 = 4$$
$$r^2 = 1$$
$$r = \pm 1$$
$$r = 1$$



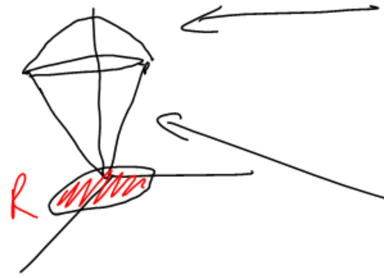
R: $0 \le r \le 1$
$0 \le \theta \le 2\pi$

$$z_{top} - z_{bottom} = 4-3r^2-r^2$$
$$= 4-4r^2$$

$$dA = r\,dr\,d\theta$$

$$V = \iint_R (z_{top} - z_{bottom})\,dA$$
$$= \int_0^{2\pi}\int_0^1 (4-4r^2)r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 (4r-4r^3)\,dr\,d\theta$$
$$= \int_0^{2\pi} \Big[2r^2 - r^4\Big]_0^1\,d\theta$$
$$= \int_0^{2\pi} 1\,d\theta$$
$$= 2\pi$$

(29)



$$x^2+y^2+z^2=a^2$$
$$z^2=a^2-r^2$$
$$z=\pm\sqrt{a^2-r^2}$$
$$z=\sqrt{a^2-r^2}$$

$$z=\sqrt{x^2+y^2}$$
$$z=r$$

Intersection: $z=z$

$$\sqrt{a^2-r^2}=r$$
$$a^2-r^2=r^2$$
$$a^2=2r^2$$
$$r^2=\frac{a^2}{2}$$
$$r=\pm\frac{a}{\sqrt{2}}$$
$$r=\frac{a}{\sqrt{2}}$$

R: $0 \le r \le \frac{a}{\sqrt{2}}$

$0 \le \theta \le 2\pi$

$$dA = r\,dr\,d\theta$$

$$V=\iint_R (z_{top}-z_{bottom})\,dA$$

$$=\int_0^{2\pi}\int_0^{\frac{a}{\sqrt{2}}}\left(\sqrt{a^2-r^2}-r\right)r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\frac{a}{\sqrt{2}}}\left(r\sqrt{a^2-r^2}-r^2\right)dr\,d\theta$$

Continued →

(29) Cont'd

$$\boxed{\begin{aligned} &\text{Sub } u = a^2 - r^2 \\ &\quad du = -2r\,dr \\ &-\tfrac{1}{2}\,du = r\,dr \\ &\int r\sqrt{a^2-r^2}\,dr = -\tfrac{1}{2}\int \sqrt{u}\,du \\ &\qquad = -\tfrac{1}{2}\cdot\tfrac{2}{3}\,u^{3/2} + C \\ &\qquad = -\tfrac{1}{3}\left(a^2-r^2\right)^{3/2} + C \end{aligned}}$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}\left(a^2 - r^2\right)^{3/2} - \frac{r^3}{3} \right]_{r=0}^{r=\frac{a}{\sqrt{2}}} d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}\left(\frac{a^2}{2}\right)^{3/2} - \frac{1}{3}\left(\frac{a}{\sqrt{2}}\right)^3 + \frac{1}{3}\left(a^2\right)^{3/2} \right] d\theta$$

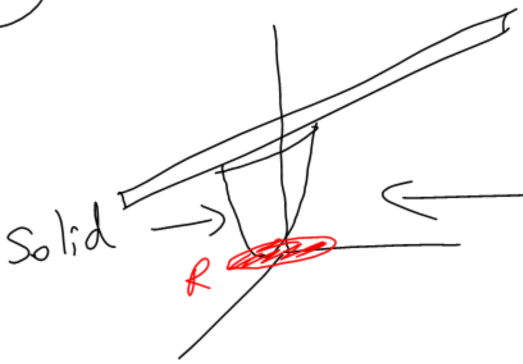
$$= \int_0^{2\pi} \left[-\frac{1}{3}\,\frac{a^3}{2\sqrt{2}} - \frac{1}{3}\,\frac{a^3}{2\sqrt{2}} + \frac{1}{3}\,a^3 \right] d\theta$$

$$= \frac{a^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\theta\Big|_0^{2\pi}$$

$$= \frac{a^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right)2\pi$$

$$= \frac{a^3\pi\left(2 - \sqrt{2}\right)}{3}$$

(37)



$z = y + 4$

$z = r\sin\theta + 4$

$z = x^2 + y^2 + y$

$= r^2 + r\sin\theta$

Intersection:

$z = z$

$r\sin\theta + 4 = r^2 + r\sin\theta$

$4 = r^2$

$r = \pm 2$

$r = 2$

$R: \quad 0 \le r \le 2$

$\quad\;\; 0 \le \theta \le 2\pi$

$dA = r\,dr\,d\theta$

$$V = \iint_R (z_{top} - z_{bottom})\, dA$$

$$= \int_0^{2\pi}\int_0^{2} (4 - r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (4r - r^3)\, dr\,d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$$

Continued $\longrightarrow$

37 Cont'd

$$= \int_0^{2\pi} 4\, d\theta$$

$$= 4\theta \Big|_0^{2\pi}$$

$$= 8\pi$$

39

$z = 12 - 2x^2 - y^2$

$z = x^2 + 2y^2$

Intersection:

$$z = z$$
$$12 - 2x^2 - y^2 = x^2 + 2y^2$$
$$12 = 3x^2 + 3y^2$$
$$12 = 3r^2$$
$$4 = r^2$$
$$r = \pm 2$$
$$r = 2$$

$R: \quad 0 \le r \le 2$
$\quad\quad 0 \le \theta \le 2\pi$

$$dA = r\,dr\,d\theta$$

$$z_{top} - z_{bottom} = 12 - 3x^2 - 3y^2$$
$$= 12 - 3r^2$$

$$V = \iint_R (z_{top} - z_{bottom})\,dA$$
$$= \int_0^{2\pi}\int_0^2 (12 - 3r^2)\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (12r - 3r^3)\,dr\,d\theta$$
$$= \int_0^{2\pi} \left[6r^2 - \frac{3r^4}{4}\right]_0^2 d\theta$$

Continued $\longrightarrow$

39 Cont'd

$$= \int_0^{2\pi} 12 \, d\theta$$

$$= 12\theta \Big|_0^{2\pi}$$

$$= 24\pi$$