⑤ $f = xy$ $\quad \underbrace{4x^2 + 9y^2}_{g} = 36$

$\nabla f = [y, x]$

$\nabla g = [8x, 18y]$

$\nabla f = \lambda \nabla g$

$[y, x] = \lambda [8x, 18y]$

(1) $y = \lambda 8x \Rightarrow \lambda = \frac{y}{8x}$

(2) $x = \lambda 18y \Rightarrow \lambda = \frac{x}{18y}$

(3) $4x^2 + 9y^2 = 36$

$\lambda = \lambda$

$\frac{y}{8x} = \frac{x}{18y}$

$18y^2 = 8x^2$

$9y^2 = 4x^2$

$9y^2 = 4x^2 \to (3)$: $9y^2 + 9y^2 = 36$

$18y^2 = 36$

$y^2 = 2$

$y = \pm\sqrt{2}$

Continued $\longrightarrow$

(5) Cont'd $\quad y^2 = 2 \rightarrow$

$$9y^2 = 4x^2$$
$$18 = 4x^2$$
$$x^2 = \frac{9}{2}$$
$$x = \pm \frac{3}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}$$

Point	$f = xy$
$\left(-\frac{3\sqrt{2}}{2}, -\sqrt{2}\right)$	3
$\left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right)$	-3
$\left(\frac{3\sqrt{2}}{2}, -\sqrt{2}\right)$	-3
$\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right)$	3

The function achieves a maximum of $f = 3$ at $\pm\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right)$.

The function achieves a minimum of $f = -3$ at $\pm\left(\frac{3\sqrt{2}}{2}, -\sqrt{2}\right)$.

(9) $f = x + y + z$ $\qquad \underbrace{x^2 + 4y^2 + 9z^2}_{g} = 36$

$\nabla f = [1, 1, 1]$

$\nabla g = [2x, 8y, 18z]$

$\nabla f = \lambda \nabla g$

$[1, 1, 1] = \lambda [2x, 8y, 18z]$

(1) $1 = \lambda(2x) \Rightarrow \lambda = \frac{1}{2x}$

(2) $1 = \lambda(8y) \Rightarrow \lambda = \frac{1}{8y}$

(3) $1 = \lambda(18z) \Rightarrow \lambda = \frac{1}{18z}$

(4) $x^2 + 4y^2 + 9z^2 = 36$

$\lambda = \lambda = \lambda$

$\frac{1}{2x} = \frac{1}{8y} = \frac{1}{18z}$

$\frac{1}{2x} = \frac{1}{8y}$

$8y = 2x$

$y = \frac{x}{4}$

$\frac{1}{2x} = \frac{1}{18z}$

$18z = 2x$

$z = \frac{x}{9}$

Continued $\rightarrow$

(9) Cont'd

$y = \frac{x}{4}, \quad z = \frac{x}{9} \rightarrow \quad x^2 + 4y^2 + 9z^2 = 36$

$$x^2 + 4\left(\frac{x^2}{16}\right) + 9\left(\frac{x^2}{81}\right) = 36$$

$$x^2 + \frac{x^2}{4} + \frac{x^2}{9} = 36$$

$$\frac{49}{36} x^2 = 36$$

$$x^2 = \frac{1296}{49}$$

$$x = \pm \frac{36}{7}$$

Recall that $y = \frac{x}{4}$ and $z = \frac{x}{9}$.

Point	$f = x + y + z$
$\left(\frac{36}{7}, \frac{9}{7}, \frac{4}{7}\right)$	7
$\left(\frac{-36}{7}, \frac{-9}{7}, \frac{-4}{7}\right)$	-7

The function achieves a maximum of $f = 7$ at $\left(\frac{36}{7}, \frac{9}{7}, \frac{4}{7}\right)$.

The function achieves a minimum of $f = -7$ at $\left(\frac{-36}{7}, \frac{-9}{7}, \frac{-4}{7}\right)$.

(19) Minimize the distance between (x,y) and $(0,0)$.

$$\text{distance} = \sqrt{(x-0)^2 + (y-0)^2}$$
$$= \sqrt{x^2+y^2}$$

It is equivalent to minimize $\text{distance}^2 = x^2+y^2$ (because this yields the same point (x,y) as minimizing the distance.)

Let $f = x^2+y^2$ $\qquad \underbrace{3x+4y}_{g} = 100$

$$\nabla f = [2x, 2y]$$
$$\nabla g = [3,4]$$
$$\nabla f = \lambda \nabla g$$
$$[2x, 2y] = \lambda [3,4]$$

(1) $2x = \lambda(3) \Rightarrow \lambda = \frac{2x}{3}$

(2) $2y = \lambda(4) \Rightarrow \lambda = \frac{y}{2}$

(3) $3x+4y = 100$

$$\lambda = \lambda$$
$$\frac{2x}{3} = \frac{y}{2}$$

Continued →

(19) Cont'd

$$\frac{2x}{3} = \frac{y}{2}$$

$$\frac{4x}{3} = y$$

$$y = \frac{4x}{3}$$

$y = \frac{4x}{3} \rightarrow$ (3) : $\quad 3x + 4\left(\frac{4x}{3}\right) = 100$

$$\frac{25x}{3} = 100$$

$$x = 12$$

$$y = \frac{4x}{3} = 16$$

The point is $(x, y) = (12, 16)$.

(21) (Refers us to (29) Section 12.5)

Minimize the distance between (x, y, z) and $(0, 0, 0)$.

$$\text{distance} = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$= \sqrt{x^2 + y^2 + z^2}$$

It is equivalent to minimize $\text{distance}^2 = x^2 + y^2 + z^2$ (because this yields the same point (x, y, z) as minimizing the distance).

Let $f = x^2 + y^2 + z^2$ $\quad \underbrace{12x + 4y + 3z}_{g} = 169$

$$\nabla f = [2x, 2y, 2z]$$

$$\nabla g = [12, 4, 3]$$

$$\nabla f = \lambda \nabla g$$

$$[2x, 2y, 2z] = \lambda [12, 4, 3]$$

(1) $2x = \lambda(12) \Rightarrow \lambda = \frac{x}{6}$

(2) $2y = \lambda(4) \Rightarrow \lambda = \frac{y}{2}$

(3) $2z = \lambda(3) \Rightarrow \lambda = \frac{2z}{3}$

(4) $12x + 4y + 3z = 169$

(21) Cont'd

$$\lambda = \lambda = \lambda$$

$$\frac{x}{6} = \frac{y}{2} = \frac{2z}{3}$$

$$\frac{x}{6} = \frac{y}{2}$$

$$\frac{x}{3} = y$$

$$y = \frac{x}{3}$$

$$\frac{x}{6} = \frac{2z}{3}$$

$$\frac{3}{2}\left(\frac{x}{6}\right) = z$$

$$z = \frac{x}{4}$$

$y = \frac{x}{3}$, $z = \frac{x}{4} \to (4)$: $12x + 4\left(\frac{x}{3}\right) + 3\left(\frac{x}{4}\right) = 169$

$$\frac{169}{12}x = 169$$

$$x = 12$$

Recall $y = \frac{x}{3}$ and $z = \frac{x}{4}$.

The point is $(x, y, z) = (12, 4, 3)$.

23 (Refers us to 31 Section 12.5)

Minimize the distance between (x, y, z) and $(7, -7, 0)$.

$$\text{distance} = \sqrt{(x-7)^2 + (y+7)^2 + (z-0)^2}$$

$$= \sqrt{(x-7)^2 + (y+7)^2 + z^2}$$

It is equivalent to minimize

$$\text{distance}^2 = (x-7)^2 + (y+7)^2 + z^2$$

(because this yields the same point (x, y, z) as minimizing the distance.)

Let $f = (x-7)^2 + (y+7)^2 + z^2$

$$\underbrace{2x + 3y + z}_{g} = 49$$

$$\nabla f = [2(x-7), 2(y+7), 2z]$$

$$\nabla g = [2, 3, 1]$$

$$\nabla f = \lambda \nabla g$$

$$[2(x-7), 2(y+7), 2z] = \lambda [2, 3, 1]$$

1. $2(x-7) = \lambda(2) \Rightarrow \lambda = x - 7$
2. $2(y+7) = \lambda(3) \Rightarrow \lambda = \frac{2}{3}(y+7)$
3. $2z = \lambda$

4. $2x + 3y + z = 49$

Continued $\rightarrow$

23 Cont'd

$$\lambda = \lambda = \lambda$$

$$x - 7 = \frac{2}{3}(y+7) = 2z$$

$$x - 7 = \frac{2}{3}y + \frac{14}{3}$$

$$x - \frac{35}{3} = \frac{2}{3}y$$

$$\frac{2}{3}y = x - \frac{35}{3}$$

$$y = \frac{3}{2}x - \frac{35}{2}$$

$$x - 7 = 2z$$

$$\frac{x}{2} - \frac{7}{2} = z$$

$$z = \frac{x}{2} - \frac{7}{2}$$

$y = \frac{3}{2}x - \frac{35}{2}$, $z = \frac{x}{2} - \frac{7}{2} \rightarrow (4)$:

$$2x + 3\left(\frac{3}{2}x - \frac{35}{2}\right) + \frac{x}{2} - \frac{7}{2} = 49$$

$$2x + \frac{9}{2}x + \frac{x}{2} - \frac{105}{2} - \frac{7}{2} = 49$$

$$7x = 105$$

$$x = 15$$

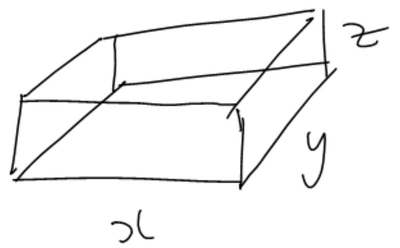
Recall $y = \frac{3}{2}x - \frac{35}{2} = 5$

$$z = \frac{x}{2} - \frac{7}{2} = 4$$

The point is $(x, y, z) = (15, 5, 4)$.

(33) (Refers us to (41) Section 12.5)

Let the dimensions of the box (in inches) be:



Note: The front and back have area xz.
The two ends " yz.
The top and bottom " xy.

Minimize the cost (in ¢)

$$f = 3(2xy) + 6(2xz) + 9(2yz)$$

$$f = 6xy + 12xz + 18yz$$

$$\underbrace{xyz}_{g} = 750$$

$$\nabla f = [6y+12z,\ 6x+18z,\ 12x+18y]$$

$$\nabla g = [yz,\ xz,\ xy]$$

$$\nabla f = \lambda \nabla g$$

$$[6y+12z,\ 6x+18z,\ 12x+18y] = \lambda [yz,\ xz,\ xy]$$

Continued $\longrightarrow$

(33) Cont'd

(1) $6y + 12z = \lambda yz \Rightarrow \lambda = \dfrac{6y+12z}{yz}$

$\Rightarrow \lambda = \dfrac{6}{z} + \dfrac{12}{y}$

(2) $6x + 18z = \lambda xz \Rightarrow \lambda = \dfrac{6}{z} + \dfrac{18}{x}$

(3) $12x + 18y = \lambda xy \Rightarrow \lambda = \dfrac{12}{y} + \dfrac{18}{x}$

(4) $xyz = 750$

$$\lambda = \lambda = \lambda$$

$$\underbrace{\frac{6}{z} + \frac{12}{y} = \frac{6}{z} + \frac{18}{x}} = \frac{12}{y} + \frac{18}{x}$$

$$\frac{12}{y} = \frac{18}{x}$$

$$12x = 18y$$

$$\frac{2}{3}x = y$$

$$\frac{6}{z} + \frac{12}{y} = \frac{12}{y} + \frac{18}{x}$$

$$\frac{6}{z} = \frac{18}{x}$$

$$6x = 18z$$

$$\frac{x}{3} = z$$

Continued $\longrightarrow$

33 Cont'd

$y = \frac{2}{3}x, \quad z = \frac{x}{3} \rightarrow \quad xyz = 750$

$$x\left(\frac{2}{3}x\right)\left(\frac{x}{3}\right) = 750$$

$$\frac{2}{9}x^3 = 750$$

$$x^3 = 3375$$

$$x = 15$$

Recall $\quad y = \frac{2}{3}x = 10$

$$z = \frac{x}{3} = 5$$

The dimensions are $x = 15$ inches, $y = 10$ inches, $z = 5$ inches.