③ $f = e^{-x^2-y^2}$

$\nabla f = [-2x e^{-x^2-y^2}, -2y e^{-x^2-y^2}]$

$\nabla f(P) = [0, 0]$

5

$$f(x, y, z) = y^2 - z^2$$

$$\nabla f = [0, 2y, -2z]$$

$$\nabla f(P) = [0, 6, -4]$$

⑦ $f = e^x \sin y + e^y \sin z + e^z \sin x$

$\nabla f = [e^x \sin y + e^z \cos x,\ e^x \cos y + e^y \sin z,$
$e^y \cos z + e^z \sin x]$

$\nabla f(P) = [1, 1, 1]$

(11) $f = x^2 + 2xy + 3y^2$

$\nabla f = [2x + 2y,\ 2x + 6y]$

$\nabla f(P) = [6,\ 10]$

direction $= [1, 1]$

$\vec{u} = \frac{1}{\sqrt{2}}[1, 1]$

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$= \frac{1}{\sqrt{2}}(16)$

$= \frac{16}{\sqrt{2}}$ or $8\sqrt{2}$

(13) $f = x^3 - x^2 y + xy^2 + y^3$

$\nabla f = [3x^2 - 2xy + y^2, \; -x^2 + 2xy + 3y^2]$

$\nabla f(P) = [6, 0]$

$\text{direction} = 2\bar{i} + 3\vec{j}$

$= [2, 3]$

$\bar{u} = \frac{1}{\sqrt{13}} [2, 3]$

$D_{\bar{u}} f = \nabla f \cdot \bar{u}$

$= \frac{12}{\sqrt{13}}$ or $\frac{12\sqrt{13}}{13}$

(19) $f = e^{xyz}$

$\nabla f = [yze^{xyz}, xze^{xyz}, xye^{xyz}]$

$\nabla f(P) = [0, -12, 0]$

direction $= \vec{j} - \vec{k}$

$= [0, 1, -1]$

$\bar{u} = \frac{1}{\sqrt{2}}[0, 1, -1]$

$D_{\bar{u}} f = \nabla f \cdot \vec{u}$

$= \frac{-12}{\sqrt{2}}$ or $-6\sqrt{2}$

(21) $f = 2x^2 + 3xy + 4y^2$

$\nabla f = [4x + 3y,\ 3x + 8y]$

$\nabla f(P) = [7, 11]$ $\qquad \|\nabla f(P)\| = \sqrt{170}$

The maximum rate of increase of f at point P is $\sqrt{170}$ $\dfrac{\text{units of } f}{\text{units of } \sqrt{x^2+y^2}}$.

It occurs in direction $[7, 11]$.

(25) $f = 3x^2 + y^2 + 4z^2$

$\nabla f = [6x, 2y, 8z]$

$\nabla f(P) = [6, 10, -16]$

$\| \nabla f(P) \| = \sqrt{392}$

The maximum rate of increase of f at point P is $\sqrt{392}$ $\dfrac{\text{units of } f}{\text{units of } \sqrt{x^2+y^2+z^2}}$.

It occurs in direction $[6, 10, -16]$.

(45) $w = 10 + xy + xz + yz$

$\nabla w = [y+z, x+z, x+y]$

$\nabla w(P) = [5, 4, 3]$

direction $= [3,4,4] - [1,2,3]$

$= [2,2,1]$

$\bar{u} = \frac{1}{\sqrt{9}} [2,2,1]$

$= \frac{1}{3} [2,2,1]$

$D_{\bar{u}} w = \nabla w \cdot \bar{u}$

$= \frac{21}{3}$

$= 7 \quad \frac{°C}{km}$

$\frac{dw}{dt} = \frac{dw}{ds} \frac{ds}{dt}$

$= 7 \frac{°C}{km} \left(2 \frac{km}{min}\right)$

$= 14 \frac{°C}{min}$

(47) $W = 50 + xyz$

a) $\nabla W = [yz, xz, xy]$

$\nabla W(P) = [4, 3, 12]$

direction $= [1, 2, 2]$

$\bar{u} = \frac{1}{\sqrt{9}} [1, 2, 2]$

$= \frac{1}{3} [1, 2, 2]$

$D_{\vec{u}} W = \nabla W \cdot \vec{u}$

$= \frac{34}{3} \quad \frac{°C}{ft}$

b) $\nabla W(P) = [4, 3, 12]$

$\| \nabla W(P) \| = \sqrt{169} = 13$

The maximum rate of increase of W at point P is $13 \frac{°C}{ft}$.

It occurs in direction $[4, 3, 12]$.

(51) $z^2 = x^2 + y^2$

$$\underbrace{x^2 + y^2 - z^2}_{F} = 0$$

$$\nabla F = [2x, 2y, -2z]$$

$$\nabla F(P) = [6, 8, 10]$$

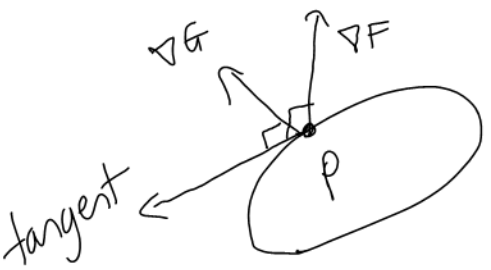
This is a normal vector to the cone (and therefore to the ellipse) at point P.

$$\underbrace{2x + 3y + 4z + 2}_{G} = 0$$

$$\nabla G = [2, 3, 4]$$

$$\nabla G(P) = [2, 3, 4]$$

This is a normal vector to the plane (and therefore to the ellipse) at point P.



tangent to ellipse $= \nabla F \times \nabla G$

$$\boxed{\begin{matrix} 6 & 8 & 10 & 6 & 8 \\ 2 & 3 & 4 & 2 & 3 \end{matrix}} \qquad = [2, -4, 2]$$

Continued →

(51) Cont'd

The tangent vector is a normal vector to the normal plane.

Let $\vec{n} = [2, -4, 2]$

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

$$\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$$

$$2x - 4y + 2z = -20$$

or $$x - 2y + z = -10$$

or $$z = -x + 2y - 10$$