9.9 Power Series Representations

Recall $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $-1<r<1$

$\Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $-1<x<1$

and $\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$ for $-1<x<1$

FACT

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ and } \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n \text{ for } -1<x<1}$$

Ex: Find a power series for $\frac{1}{1+x^3}$ centred at $c=0$.

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n$$

or $\sum_{n=0}^{\infty} (-1)^n x^{3n}$

Note: Interval of convergence is $-1<x<1$

Ex: Find a power series for $\frac{10}{3+2x}$ centred at $c=1$.

$$\frac{10}{3+2x} = \frac{10}{? + 2(x-1)}$$

$$= \frac{10}{5+2(x-1)}$$

$$= \frac{10}{5}\left[\frac{1}{1+\frac{2}{5}(x-1)}\right]$$

$$= 2\sum_{n=0}^{\infty}\left[-\frac{2}{5}(x-1)\right]^n$$

$$\text{or } 2\sum_{n=0}^{\infty}\left(\frac{-2}{5}\right)^n(x-1)^n$$

Note: Interval of convergence is $\frac{-3}{2} < x < \frac{7}{2}$

Ex: Find a power series for

$\frac{1}{(1-x)^2}$ centred at $c=0$.

Notice $\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right]$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{d}{dx}\left[\frac{1}{1-x}\right] \\
&= \frac{d}{dx} \sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} n x^{n-1} \\
&= 0 + 1 + 2x + 3x^2 + \dots
\end{aligned}$$

Note: Interval of convergence is $-1 < x < 1$

Ex: Find a power series for $\ln(1+x)$ centred at $c=0$.

Notice $\ln(1+x)+C = \int \frac{1}{1+x}\,dx$.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

$$\ln(1+x)+C = \int \frac{1}{1+x}\,dx$$

$$\ln(1+x)+C = \int \sum_{n=0}^{\infty} (-x)^n\,dx$$

$$\ln(1+x)+C = \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx$$

$$\ln(1+x)+C = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

Sub $x=0$: $\quad C = 0$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

Note: Interval of Convergence is $-1 < x \leq 1$

Ex: Find a power series for $\arctan x$ centred at $c=0$.

Notice $\arctan x + C = \int \frac{1}{1+x^2}\,dx$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\arctan x + C = \int \frac{1}{1+x^2}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Sub $x=0$: $0 + C = 0$

$C = 0$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Note: Interval of convergence is $-1 \le x \le 1$

Ex: a) Approximate $\int_0^{0.5} \frac{1}{1+x^5}\,dx$

using three terms of a power series

$$\frac{1}{1+x^5} = \sum_{n=0}^{\infty} (-1)^n (x^5)^n \quad \text{for } -1<x<1$$

$$= 1 - x^5 + x^{10} - \dots$$

$$\int_0^{0.5} \frac{1}{1+x^5}\,dx \approx \int_0^{0.5} (1-x^5+x^{10})\,dx$$

$$\approx \left[x - \frac{x^6}{6} + \frac{x^{11}}{11}\right]_0^{0.5}$$

$$\approx 0.5 - \frac{(0.5)^6}{6} + \frac{(0.5)^{11}}{11}$$

$$\approx 0.49744022$$

b) Find an upper bound for the error

$0.5 - \frac{(0.5)^6}{6} + \frac{(0.5)^{11}}{11} - \dots$ is an alternating series

$$|R_N| \le a_{N+1}$$

absolute value of next term

$$\text{Error} \le \frac{(0.5)^{16}}{16}$$

$$\le 9.6 \times 10^{-7}$$

c) Conclusion about the integral?

$$0.49744022 - 9.6\times10^{-7} \le \int_0^{0.5} \frac{1}{1+x^5}\,dx \le 0.49744022 + 9.6\times10^{-7}$$