9.8 Power Series

Power series centred at c:

$$\sum_{n=0}^{\infty} b_n (x-c)^n = b_0 + b_1(x-c) + b_2(x-c)^2 + \dots$$

Note: $b_0, b_1, b_2, \dots$ are the coefficients

FACT
For any power series one of the following is true:

1) There is a number $R>0$ called the <u>radius of convergence</u> such that the series converges absolutely if $|x-c|<R$ and diverges if $|x-c|>R$

2) The series converges absolutely for all x (we say $R=\infty$)

3) The series converges only for $x=c$ (we say $R=0$)

Convergence or divergence at $|x-c|=R$ is analyzed case by case.

The set of x-values for which the series converges is called the <u>interval of convergence</u>.

Ex: Find the interval of convergence:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n \cdot 5^n}$$

Use ratio test

Let $a_n = \frac{(-1)^n (x-3)^n}{n \cdot 5^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(x-3)^n}\right|$$

$$= \left|\frac{(x-3)}{5} \frac{n}{n+1}\right|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x-3}{5}\right|$$

Series converges if $\left|\frac{x-3}{5}\right| < 1$

$|x-3| < 5$

centre of series $c=3$

radius of convergence $R=5$

$-5 < x-3 < 5$

$-2 < x < 8$

Analyze $x=-2$ and $x=8$

$x = -2$: Series $= \sum_{n=1}^{\infty} \frac{(-1)^n (-5)^n}{n \cdot 5^n}$

$= \sum_{n=1}^{\infty} \frac{5^n}{n \cdot 5^n}$

$= \sum_{n=1}^{\infty} \frac{1}{n}$

Diverges (p-Series)

$x = 8$: Series $= \sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{n \cdot 5^n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Converges (Alternating Series Test)

Interval of Convergence:

$-2 < x \leq 8$ or $(-2, 8]$

Ex: Find the interval of Convergence:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$= \left|\frac{x}{n+1}\right|$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$$

Series converges for all x $(R=\infty)$

Interval of convergence : $(-\infty, \infty)$

Ex: Find the interval of convergence :

$$\sum_{n=0}^{\infty} n!\,(x-2)^n$$

If $x\neq 2$: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\,(x-2)^{n+1}}{n!\,(x-2)^n}\right|$

$= |(n+1)(x-2)|$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$$

Series diverges

If $x=2$: Series $= 1+0+0+0+\ldots$

Series converges

Radius of Convergence $R=0$

Interval of Convergence $x=2$

FACT

Suppose $f(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$

has radius of convergence $R > 0$.

Then $f'(x)$ and $\int f(x)dx$ have radius of convergence $= R$.

On the interval $(c-R, c+R)$:

$$f'(x) = \sum_{n=0}^{\infty} n b_n (x-c)^{n-1}$$

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{b_n (x-c)^{n+1}}{n+1} + C_1$$

Ex: Given $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ has $R=1$.

Find $f'(x)$ and $\int f(x)dx$ on the interval $(-1,1)$:

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n} = \sum_{n=1}^{\infty} x^{n-1}$$

$$\int f(x)dx = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} + C_1$$