9.7 Taylor and Maclaurin Polynomials

The <u>n^{th} degree Taylor polynomial of f centred at c</u> is:

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $c=0$ then it's called the <u>n^{th} degree Maclaurin polynomial of f</u>.

<u>Ex</u>: Find the 7^{th} degree Maclaurin polynomial of $f(x) = \sin x$

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = 1$
$f^{(6)}(x) = -\sin x$	$f^{(6)}(0) = 0$
$f^{(7)}(x) = -\cos x$	$f^{(7)}(0) = -1$

$$P_7(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(7)}(0)}{7!}x^7$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

<u>Ex</u>: Find the n^{th} degree Taylor polynomial of $f(x) = \ln x$ centred at $c = 1$.

$f(x) = \ln x$	$f(1) = 0$
$f'(x) = \frac{1}{x}$	$f'(1) = 1$
$f''(x) = -x^{-2}$	$f''(1) = -1$
$f'''(x) = 2x^{-3}$	$f'''(1) = 2$
$f^{(4)}(x) = -6x^{-4}$	$f^{(4)}(1) = -6$
$\vdots$	
$f^{(n)}(x) = (-1)^{n+1}(n-1)!\,x^{-n}$	$f^{(n)}(1) = (-1)^{n+1}(n-1)!$

$$P_n(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \dots + \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= x - 1 - \frac{1}{2!}(x-1)^2 + \dots + \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$$

$$= x - 1 - \frac{1}{2}(x-1)^2 + \dots + \frac{(-1)^{n+1}}{n}(x-1)^n$$

$$f(x) = P_n(x) + \underbrace{R_n(x)}_{\text{remainder or error}}$$

FACT

$$\boxed{R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \quad \text{where } z \text{ is a number between } x \text{ and } c}$$

<u>Ex</u> a) Find the 2^{nd} degree Taylor polynomial of $f(x) = \sqrt{x}$ centred at $c=4$

$f(x) = \sqrt{x}$ $\qquad f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\qquad f''(4) = -\frac{1}{4}\left(\frac{1}{8}\right) = -\frac{1}{32}$

$$P_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

b) Use it to approximate $\sqrt{4.1}$

$$P_2(4.1) = 2 + \frac{1}{4}(0.1) - \frac{1}{64}(0.1)^2$$

$$\approx 2.02484375$$

c) Find an upper bound for $|R_2(4.1)|$

$$f'''(x) = -\frac{1}{4}\left(\frac{-3}{2}\right)x^{-5/2}$$
$$= \frac{3}{8}x^{-5/2}$$

$$|R_2(x)| = \left|\frac{f^{(3)}(z)}{3!}(x-c)^3\right|$$

$$|R_2(4.1)| = \frac{1}{6}\,\frac{3}{8}\,z^{-5/2}\,(4.1-4)^3$$

for z between 4 and 4.1

$$\leq \frac{1}{16}(4)^{-5/2}(0.1)^3$$

$$\leq 0.000002$$

d) Approximate $\sqrt{4.1}$, with error

$$2.02484375 - 0.000002 \leq \sqrt{4.1} \leq 2.02484375 + 0.000002$$

Ex: Find N so that the Maclaurin polynomial $P_N(x)$ approximates e^{-1} with error less than 0.001

$$f(x) = e^x$$
$$f^{(n)}(x) = e^x$$

Use $x = -1$ and $c = 0$.

$$|R_N(x)| = \left| \frac{f^{(N+1)}(z)}{(N+1)!} (x-c)^{N+1} \right|$$

for z between -1 and 0

$$= \left| \frac{e^z}{(N+1)!} (-1)^{N+1} \right|$$

$$= \frac{e^z}{(N+1)!}$$

$$\leq \frac{1}{(N+1)!}$$

$$\leq 0.001$$

Guess and check $\quad N=5$ ✗ $\quad N=6$ ✓

$$\boxed{N \geq 6}$$