9.6 The Ratio and Root Tests

Ratio Test

Consider $\sum_{n=1}^{\infty} a_n$. Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

If $L < 1$ then the series converges absolutely.

$L = 1$ the test is inconclusive.

$L > 1$ the series diverges.

Ex: Test for Convergence:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{3^n}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3} \left(\frac{n+1}{n} \right)^3$$

$$= \frac{1}{3} \left(1 + \frac{1}{n} \right)^3$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3}$$

The series converges absolutely.

Recall: $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$ (Section 5.6)

Ex: Test for convergence:

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \frac{(n+1)^{n+1}}{(n+1)\cdot n!} \cdot \frac{n!}{n^n}$$

$$= \frac{(n+1)^n}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n$$

$$= \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$= e$$

The series diverges.

Root Test

Consider $\sum_{n=1}^{\infty} a_n$. Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

If $L < 1$ then the series converges absolutely.

$L = 1$ the test is inconclusive.

$L > 1$ the series diverges.

Ex: Test for convergence:

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+5}\right)^n$$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{2n+3}{3n+5}\right)^n}$$

$$= \left(\frac{2n+3}{3n+5}\right)$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{2n+3}{3n+5}\right)$$

$$= \frac{2}{3} \quad \text{by L'Hôpital's Rule.}$$

The series converges absolutely.