9.5 Alternating Series

In an alternating series, signs alternate.

Examples:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

Alternating Series Test

Let $a_n > 0$. The alternating series

$\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ both converge

if $\lim_{n\to\infty} a_n = 0$ <u>and</u> $a_{n+1} \le a_n$ for all n.

<u>Ex</u>: Test $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ for convergence.

It's an alternating series. ✓

$$\lim_{n\to\infty} \frac{1}{2n-1} = 0 \checkmark$$

$$a_{n+1} = \frac{1}{2(n+1)-1} = \frac{1}{2n+1}$$

$$\frac{1}{2n+1} \le \frac{1}{2n-1} \text{ for all } n \checkmark$$

The series converges.

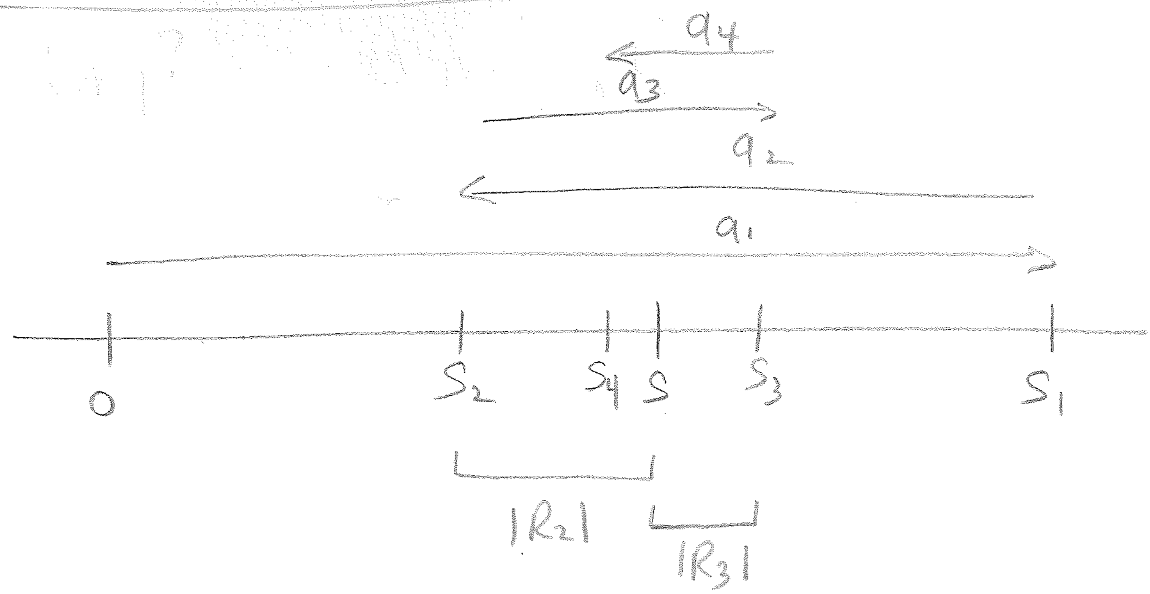
Recall:

partial sum S_N = sum of first N terms

S = sum of series

remainder/error $R_N = S - S_N$

FACT
If a series converges by the Alternating Series Test
then $|R_N| \le a_{N+1}$.



$|R_2| \le a_3$

$|R_3| \le a_4$

etc.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ Converges by the Alternating Series Test.

a) Calculate S_{19}

$$S_{19} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{19}$$

$$\approx 0.7188$$

b) Find an upper bound for $|R_{19}|$

$$a_{20} = \frac{1}{20} = 0.05$$

$$|R_{19}| \leq 0.05$$

c) Estimate S

$$0.7188 - 0.05 \leq S \leq 0.7188 + 0.05$$

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}+1}$ converges by the Alternating Series Test.

Find N so that $|R_N| \leq 0.005$

Plan: Solve $a_{N+1} \leq 0.005$

Then $|R_N| \leq a_{N+1} \leq 0.005$

$$a_{N+1} \leq 0.005$$

$$\frac{1}{\sqrt{N+1}+1} \leq 0.005$$

$$\frac{1}{0.005} \leq \sqrt{N+1} + 1$$

$$199 \leq \sqrt{N+1}$$

$$39601 \leq N+1$$

$$N \geq 39600$$

Def

Let $\sum_{n=1}^{\infty} a_n$ be any series (not necessarily alternating).

$\sum_{n=1}^{\infty} a_n$ <u>converges absolutely</u>

if $\sum_{n=1}^{\infty} |a_n|$ converges.

$\sum_{n=1}^{\infty} a_n$ <u>converges conditionally</u>

if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

<u>Ex</u>: a) $1 - \frac{1}{2} + \frac{1}{3} - \dots$ converges

$1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges

$\sum \frac{(-1)^{n+1}}{n}$ <u>converges conditionally</u>

b) $1 - \frac{1}{4} + \frac{1}{9} - \dots$ converges

$1 + \frac{1}{4} + \frac{1}{9} + \dots$ converges

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ <u>converges absolutely</u>

Absolute Convergence Theorem

Let $\sum_{n=1}^{\infty} a_n$ be any series (not necessarily alternating).

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Ex: $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$ converges.

Is it conditionally convergent or absolutely convergent?

Let $a_n = \frac{|\cos n|}{n^3}$ $\quad b_n = \frac{1}{n^3}$

Note: $\cos x = 0$ for $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$\cos(\text{integer}) \neq 0$

$0 < \frac{|\cos n|}{n^3} \leq \frac{1}{n^3}$ for $n \geq 1$ ✓

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series)

$\sum_{n=1}^{\infty} \frac{|\cos n|}{n^3}$ converges by Direct Comparison Test.

$\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$ converges absolutely.