9.4 Comparisons of Series

Direct Comparison Test

Suppose $0 < a_n \le b_n$ for all n.

If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

Also true if index starts at $n=0$ etc.

Recall: $\sum_{n=0}^{\infty} ar^n$ converges if $-1 < r < 1$
diverges if $|r| \ge 1$

Recall: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$
diverges if $p \le 1$

Ex: Test $\sum_{n=1}^{\infty} \frac{1+\ln n}{n}$ for convergence

$$\frac{1+\ln n}{n} \geq \frac{1}{n} \quad \text{for } n \geq 1$$

So $0 < \frac{1}{n} \leq \frac{1+\ln n}{n}$ for $n \geq 1$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series)

$\Rightarrow \sum_{n=1}^{\infty} \frac{1+\ln n}{n}$ diverges

Ex: Test $\sum_{n=0}^{\infty} \frac{1}{2^{(n^2)}}$ for convergence

$$n^2 \geq n \quad \text{for } n \geq 0$$

$$2^{(n^2)} \geq 2^n \quad \text{for } n \geq 0$$

$$\frac{1}{2^{(n^2)}} \leq \frac{1}{2^n} \quad \text{for } n \geq 0$$

So $0 < \frac{1}{2^{(n^2)}} \leq \frac{1}{2^n}$ for $n \geq 0$

$\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges (geometric)

$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{2^{(n^2)}}$ converges

Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ for all n.

If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $0 < L < \infty$

then $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$

<u>both converge</u> or <u>both diverge</u>.

<u>Ex</u>: Test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+5}$ for convergence

Compare it with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

$\frac{1}{3n+5} > 0$ and $\frac{1}{n} > 0$ for all $n \geq 1$ ✓

$$\lim_{n\to\infty} \frac{\left(\frac{1}{3n+5}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{n}{3n+5}$$

$$= \lim_{n\to\infty} \frac{1}{3} \quad \text{(L'Hôpital's Rule)}$$

$$= \frac{1}{3}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+5}$ diverges.

Ex: Test $\sum_{n=1}^{\infty} \frac{3\sqrt{n}+2}{5n^2+3n+1}$ for convergence.

Look at dominant terms: $\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

Compare it with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$\frac{3\sqrt{n}+2}{5n^2+3n+1} > 0$ and $\frac{1}{n^{3/2}} > 0$ for $n \geq 1$ ✓

$$\lim_{n\to\infty} \frac{\left(\frac{3\sqrt{n}+2}{5n^2+3n+1}\right)}{\left(\frac{1}{n^{3/2}}\right)}$$

$$= \lim_{n\to\infty} \frac{3n^2 + 2n^{3/2}}{5n^2+3n+1}$$

$$= \lim_{n\to\infty} \frac{3 + \frac{2}{n^{1/2}}}{5 + \frac{3}{n} + \frac{1}{n^2}}$$

$$= \frac{3}{5}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{3\sqrt{n}+2}{5n^2+3n+1}$ converges