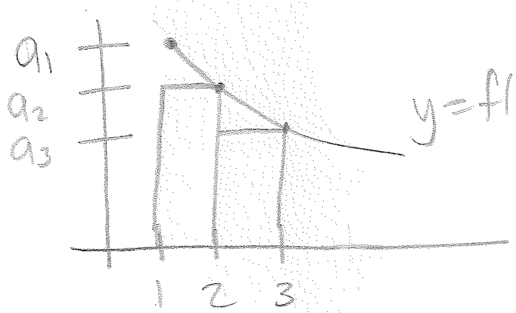
9.3 The Integral Test and p-Series

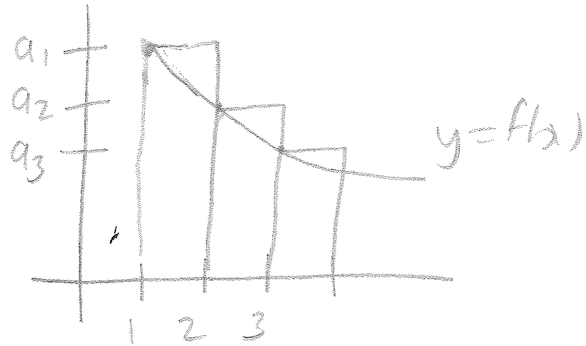
Integral Test

If f is continuous, positive and decreasing over $[1,\infty)$ and $a_n = f(n)$ for $n = 1, 2, 3, \ldots$ then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$

both converge or both diverge.

Why?





$$a_2 + a_3 + \ldots \leq \int_1^{\infty} f(x)\,dx \leq a_1 + a_2 + a_3 + \ldots$$

$\sum_{n=1}^{\infty} a_n$ is a finite real number

$\Leftrightarrow$ $\int_1^{\infty} f(x)\,dx$ is a finite real number

Ex: Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

Let $f(x) = \frac{1}{x^2}$

$f(x)$ is continuous on $[1, \infty)$ ✓

" positive " ✓

$f'(x) = -2x^{-3}$

$f'(x) < 0$ on $[1, \infty)$

$f(x)$ is decreasing on $[1, \infty)$ ✓

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} -x^{-1} \Big|_1^b$$

$$= \lim_{b\to\infty} -\frac{1}{b} + 1$$

$$= 1$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Caution: No info about the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$

It's known that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Ex: Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?

Let $f(x) = \frac{1}{x}$

$f(x)$ is continuous on $[1, \infty)$ ✓

" positive " ✓

$f'(x) = -x^{-2} < 0$ on $[1, \infty)$

$f(x)$ is decreasing on $[1, \infty)$ ✓

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \ln x \Big|_1^b$$

$$= \lim_{b\to\infty} \ln b$$

$$= \infty$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Note: $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series.

FACT

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$

Note: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series.

Def

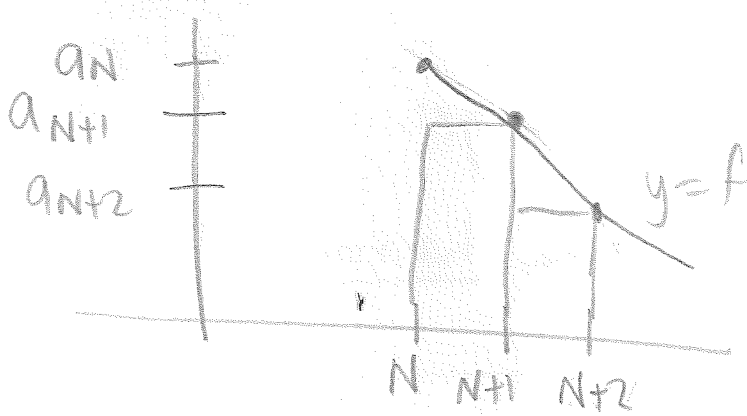
$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{N} a_n}_{\substack{\text{partial sum} \\ S_N}} + \underbrace{\sum_{n=N+1}^{\infty} a_n}_{\substack{\text{remainder or error} \\ R_N}}$$

FACT

Suppose $\sum_{n=1}^{\infty} a_n$ converges by the Integral Test.

Then $R_N \leq \int_N^{\infty} f(x)\,dx$

Why?



Ex: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the Integral Test.

a) Find S_{10}

$$S_{10} = 1 + \frac{1}{8} + \frac{1}{27} + \dots + \frac{1}{1000}$$
$$\approx 1.19753199$$

b) Find an upper bound for the error R_{10}

$$R_{10} \le \int_{10}^{\infty} \frac{1}{x^3}\,dx$$
$$\le \lim_{b\to\infty} \int_{10}^{b} \frac{1}{x^3}\,dx$$
$$\le \lim_{b\to\infty} -\frac{1}{2}x^{-2}\Big|_{10}^{b}$$
$$\le \lim_{b\to\infty} -\frac{1}{2}b^{-2} + \frac{1}{2}(10)^{-2}$$
$$\le 0.005$$

c) Estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ using S_{10} and R_{10}

$$S_{10} \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le S_{10} + R_{10}$$
$$1.19753199 \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le 1.19753199 + 0.005$$

d) Find N so that $R_N \le 0.0005$

Plan: Calculate $\int_N^\infty \frac{1}{x^3}\,dx$

Solve $\int_N^\infty \frac{1}{x^3}\,dx \le 0.0005$

$\Rightarrow \quad R_N \le 0.0005$

$$\int_N^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_N^b \frac{1}{x^3}\,dx$$

$$= \lim_{b\to\infty} -\frac{1}{2}x^{-2}\Big|_N^b$$

$$= \frac{1}{2}N^{-2}$$

Let $\frac{1}{2}N^{-2} \le 0.0005$

$$\frac{1}{N^2} \le 0.001$$

$$\frac{1}{0.001} \le N^2$$

$$\sqrt{\frac{1}{0.001}} \le N$$

$$N \geq 31.6$$

$$N \geq 32$$