9.2 Series and Convergence

Sequence: $a_1, a_2, a_3, \dots$

Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$

The N^{th} partial sum of a series is the sum of the first N terms.

Ex: Find the partial sums S_1, S_2, S_3 for $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$S_1 = \frac{1}{2}$$
$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

Notation: Let $S = \lim_{N \to \infty} S_N$ (if it exists)

Def: If $S = \lim_{N \to \infty} S_N$ exists and is a real number then the series converges to S. Otherwise the series diverges.

A <u>geometric series</u> is:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots$$

FACT

$$\boxed{\begin{array}{l} \displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } -1<r<1 \\ \displaystyle\sum_{n=0}^{\infty} ar^n \text{ diverges if } |r| \geq 1 \end{array}}$$

<u>Ex</u>: let $-1<r<1$. Consider $\sum_{n=0}^{\infty} ar^n$.
Show that $S = \frac{a}{1-r}$

$$S_N = a + ar + \ldots + ar^{N-1}$$

$$rS_N = ar + \ldots + ar^{N-1} + ar^N$$

$$S_N - rS_N = a - ar^N$$

$$S_N(1-r) = a - ar^N$$

$$S_N = \frac{a - ar^N}{1-r}$$

$$S = \lim_{N\to\infty} S_N$$

$$= \frac{a}{1-r}$$

Ex: Find the sum or state that it diverges.

a) $\sum_{n=0}^{\infty} \frac{3}{4^n}$

$= 3 + \frac{3}{4} + \frac{3}{16} + \dots$

$a = 3 \quad r = \frac{1}{4}$

$S = \frac{a}{1-r} = \frac{3}{\left(\frac{3}{4}\right)} = 4$

b) $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

$= 1 + \frac{3}{2} + \frac{9}{4} + \dots$

$r = \frac{3}{2}$ Series diverges.

c) $\sum_{n=2}^{\infty} \frac{7(3^{n-1})}{5^n}$

$= \frac{21}{25} + \frac{63}{125} + \dots$

$a = \frac{21}{25} \quad r = \frac{3}{5}$

$S = \frac{a}{1-r} = \frac{\left(\frac{21}{25}\right)}{\left(\frac{2}{5}\right)} = \frac{21}{25} \cdot \frac{5}{2} = \frac{21}{10}$

d) $\sum_{n=k}^{\infty} ar^n$ where $-1<r<1$

$= ar^k + ar^{k+1} + \ldots$

1st term $= ar^k$

ratio $= r$

$$S = \frac{1\text{st term}}{1-\text{ratio}} = \frac{ar^k}{1-r}$$

FACT

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge

then $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

and $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$ for constant c.

Ex: Find $\sum_{n=2}^{\infty} \frac{12 + 4(2)^{n+1}}{5^n}$

$$= \underbrace{\sum_{n=2}^{\infty} \frac{12}{5^n}}_{\text{geometric}} + \underbrace{\sum_{n=2}^{\infty} \frac{4(2)^{n+1}}{5^n}}_{\text{geometric}}$$

$a = \frac{12}{25}$ $r = \frac{1}{5}$ $\qquad$ $a = \frac{32}{25}$ $r = \frac{2}{5}$

$$= \frac{\left(\frac{12}{25}\right)}{\left(\frac{4}{5}\right)} + \frac{\left(\frac{32}{25}\right)}{\left(\frac{3}{5}\right)}$$

$$= \frac{12}{25} \cdot \frac{5}{4} + \frac{32}{25} \cdot \frac{5}{3}$$

$$= \frac{3}{5} + \frac{32}{15}$$

$$= \frac{41}{15}$$

A series of the form

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) = (b_1 - b_2) + (b_2 - b_3) + \ldots$$

is a <u>telescoping series</u>

Partial sums:

$$S_1 = b_1 - b_2$$
$$S_2 = b_1 - b_3$$
$$S_3 = b_1 - b_4$$
$$S_N = b_1 - b_{N+1}$$

FACT

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 - \lim_{n \to \infty} b_n$$

if the limit exists and is a real number. Otherwise the series diverges.

Ex: Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Use partial fractions to write as a telescoping series.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$= \frac{1}{n} - \frac{1}{n+1} \qquad \text{(check)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \lim_{n\to\infty} \frac{1}{n}$$

$$= 1$$

FACT: n^{th} Term Test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: Consider $\sum_{n=1}^{\infty} \frac{3n+1}{5n+1}$

$$\lim_{n\to\infty} \frac{3n+1}{5n+1} = \frac{3}{5} \neq 0$$

The series diverges.

Caution: If $\lim_{n\to\infty} a_n = 0$ the series may or may not converge.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\lim_{n\to\infty} a_n = 0$.