9.10 Taylor and Maclaurin Series

The <u>Taylor series of f centred at c</u>:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If $c=0$ then it's called the <u>Maclaurin series of f</u>.

<u>Ex</u>: Find the Maclaurin series of $f(x)=\sin x$

$f(x)=\sin x$ $\quad f(0)=0$

$f'(x)=\cos x$ $\quad f'(0)=1$

$f''(x)=-\sin x$ $\quad f''(0)=0$

$f'''(x)=-\cos x$ $\quad f'''(0)=-1$

$f^{(4)}(x)=\sin x$ $\quad f^{(4)}(0)=0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Note: Interval of convergence is $-\infty < x < \infty$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{on } -\infty < x < \infty$$

Ex: Find the Maclaurin series of
$f(x) = (1+x)^k$ where k is a real number.

$f(x) = (1+x)^k$ $\qquad$ $f(0) = 1$

$f'(x) = k(1+x)^{k-1}$ $\qquad$ $f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$ $\qquad$ $f''(0) = k(k-1)$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!} = 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

Note: Interval of convergence is $-1 < x < 1$.

Ex: Find the Maclaurin series of $f(x) = x^3 \sin x^2$, given $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$x^3 \sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(2n+1)!}$$

Ex: Find the first three nonzero terms of the Maclaurin series for $e^x\sqrt{1+x}$. Given:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

$$e^x(1+x)^{1/2} = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)\left(1 + \frac{x}{2} - \frac{x^2}{8} + \dots\right)$$

$$\begin{array}{llll} = & 1 & + \frac{x}{2} & -\frac{x^2}{8} \\ & & x & +\frac{x^2}{2} \\ & & & \frac{x^2}{2} \end{array}$$

$$= 1 + \frac{3x}{2} + \frac{7x^2}{8} + \dots$$

Ex: a) Approximate $\int_0^{0.5} e^{-x^2}\,dx$ using four nonzero terms.

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}$$

$$\int_0^{0.5} e^{-x^2}\,dx \approx \int_0^{0.5} \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}\right) dx$$

$$\approx \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right]_0^{0.5}$$

$$\approx 0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \frac{0.5^7}{42}$$

$$\approx 0.46127232$$

b) Find an upper bound for the error

$0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \frac{0.5^7}{42}$ is an alternating series

$$|R_N| \le a_{N+1}$$

absolute value of next term

Next term of e^x : $\frac{x^4}{24}$

e^{-x^2} : $\frac{x^8}{24}$

Next term of $\int_0^{0.5} e^{-x^2}\,dx$: $\left.\dfrac{x^9}{9(24)}\right|_0^{0.5}$

$$= \frac{0.5^9}{9(24)}$$

$$\approx 9.1\times10^{-6}$$

c) Conclusion about $\int_0^{0.5} e^{-x^2}\,dx$?

$$\int_0^{0.5} e^{-x^2}\,dx = 0.46127232 \pm 9.1\times10^{-6}$$

d) How many nonzero terms are required for error ≤ 0.01 ?

$|R_N| \leq a_{N+1}$

absolute value of next term

$$\int_0^{0.5} e^{-x^2}\,dx \approx 0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \dots$$

|this term| ≤ 0.01

Two nonzero terms are required.