9.1 Sequences

A sequence is an infinite ordered list of numbers.

Notation:

$a_1, a_2, a_3, \ldots$

or $\{a_n\}_{n=1}^{\infty}$

A sequence could begin at $n=0$ or any other value.

Ex: Find the first three terms

$$\left\{ \frac{(-1)^n}{2^n} \right\}_{n=0}^{\infty}$$

$1, \frac{-1}{2}, \frac{1}{4}, \ldots$

Ex: $a_0 = 0, \quad a_1 = 1$

$a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$

Find the next three terms

$a_2 = a_1 + a_0 = 1$

$a_3 = a_2 + a_1 = 2$

$a_4 = a_3 + a_2 = 3$

$0, 1, 1, 2, 3, \ldots$

If $\lim_{n\to\infty} a_n$ exists and is a finite number then the sequence converges. Otherwise it diverges.

Ex: Find the sequence's limit, if possible.

a) $\left\{\frac{1}{3n+1}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{1}{3n+1} = 0$$

Sequence converges to 0.

b) $\{n^2\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} n^2 = \infty$$

Sequence diverges.

c) $\{(-1)^n\}_{n=1}^{\infty}$

$\lim_{n\to\infty} (-1)^n$ does not exist

Sequence diverges.

d) $\left\{\frac{3n}{\sqrt{n^2+1}}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{3n}{\sqrt{n^2+1}} = \lim_{n\to\infty} 3\sqrt{\frac{n^2}{n^2+1}}$$

$$= \lim_{n\to\infty} 3\sqrt{\frac{1}{(1+\frac{1}{n^2})}}$$

$$= 3$$

Sequence converges to 3.

We can use l'Hôpital's Rule on sequences.

<u>Ex</u>: $\lim_{n\to\infty} \frac{2n}{3n+1} = \lim_{n\to\infty} \frac{2}{3}$ (by L'Hopital's Rule)

$= \frac{2}{3}$

So $\left\{\frac{2n}{3n+1}\right\}_{n=1}^{\infty}$ converges to $\frac{2}{3}$.

<u>Squeeze Theorem</u>

If $a_n \le b_n \le c_n$ for all n

and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$

then $\lim_{n\to\infty} b_n = L$

<u>Ex</u>: Find the limit of $\left\{\frac{\cos n}{n}\right\}_{n=1}^{\infty}$

$\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$ for all $n \ge 1$

$\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} \frac{\cos n}{n} = 0$

Sequence converges to 0.

$n! = n(n-1) \cdot \ldots \cdot 2 \cdot 1$

"n factorial"

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$3! = 3 \cdot 2 \cdot 1 = 6$

$2! = 2 \cdot 1 = 2$

$1! = 1$

$0! = 1$ by definition

<u>Ex</u>: Simplify $\dfrac{(n+1)!}{(n-1)!}$

$$= \frac{(n+1)n(n-1)!}{(n-1)!}$$

$$= (n+1)n$$