# 8.8 Improper Integrals

① If $f$ is continuous over $[a, \infty)$ then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

② If $f$ is continuous over $(-\infty, b]$ then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

③ If $f$ is continuous over $(-\infty, \infty)$ then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

for any real number $c$.

An integral converges if its value is a real number. Otherwise it diverges.

The integral in ③ diverges if either integral on the right side diverges.

Ex: Evaluate or show that it diverges.

a) $\int_1^\infty \frac{1}{x}\,dx$

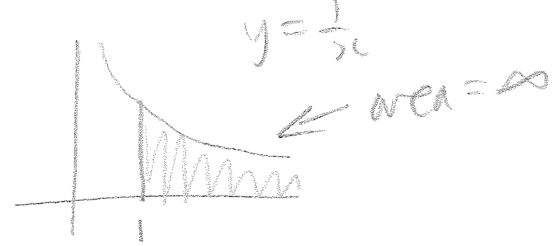
$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \ln|x| \Big|_1^b$

$= \lim_{b\to\infty} \ln b$

$= \infty$

DIVERGES



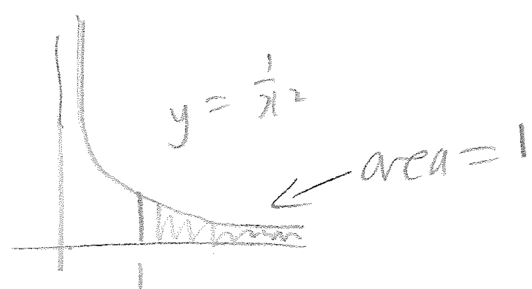

b) $\int_1^\infty \frac{1}{x^2}\,dx$

$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$

$= \lim_{b\to\infty} -x^{-1} \Big|_1^b$

$= \lim_{b\to\infty} -\frac{1}{b} + 1$

$= 1$




FACT

Let $p$ be a real number

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \text{diverges} & \text{if } p\le 1 \end{cases}$$

Ex: Evaluate or show that it diverges.

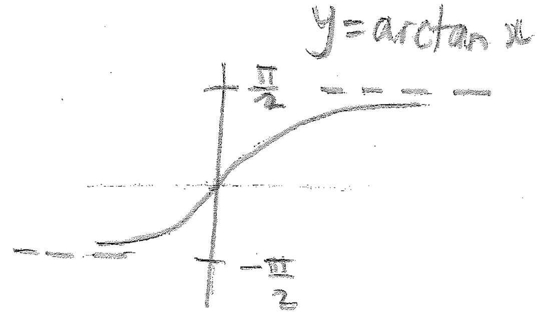
a) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$

$\displaystyle = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx$

$\displaystyle = \lim_{b\to\infty} \arctan x \Big|_0^b$

$\displaystyle = \lim_{b\to\infty} \arctan b$

$= \frac{\pi}{2}$

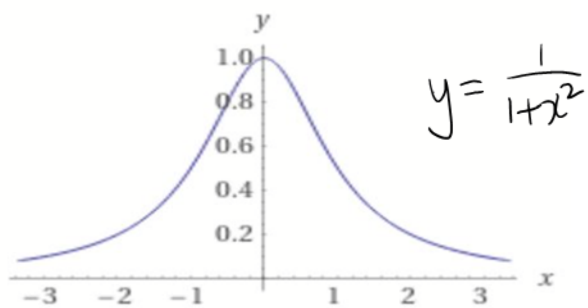



b) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

$\displaystyle = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$

$\displaystyle = 2\int_0^\infty \frac{1}{1+x^2}\,dx$

$= \pi$




c) $\displaystyle\int_0^\infty \sin x\,dx$

$\displaystyle = \lim_{b\to\infty} \int_0^b \sin x\,dx$

$\displaystyle = \lim_{b\to\infty} \left[-\cos x\right]_0^b$

$= \lim_{b\to\infty} -\cos b + 1$

Diverges ($\cos b$ oscillates as $b\to\infty$)

d) $\int_{-\infty}^{\infty} \sin x \, dx$

$= \int_{-\infty}^{0} \sin x \, dx + \int_{0}^{\infty} \sin x \, dx$

Diverges

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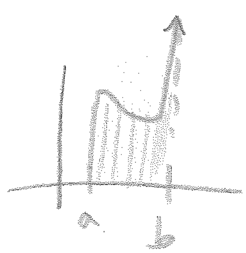
(4) Let $f$ be continuous on $[a,b)$ with an asymptote (infinite discontinuity) at $x=b$.

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

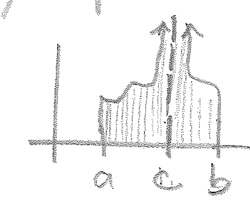



(5) Let $f$ be continuous on $(a,b]$ with an asymptote at $x=a$.

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

(6) Let $f$ be continuous on $[a,b]$ except at $x=c$ where it has an asymptote.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$




The integral in (6) diverges if either integral on the right side diverges.

<u>Ex</u>: Evaluate or show that it diverges.

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{t \to 2^+} \int_t^5 (x-2)^{-1/2}\,dx$$

$$= \lim_{t \to 2^+} 2(x-2)^{1/2} \Big|_t^5$$

$$= \lim_{t \to 2^+} 2\sqrt{3} - 2\sqrt{t-2}$$

$$= 2\sqrt{3}$$

Ex: Evaluate or show that it diverges.

$$\int_0^3 \frac{1}{(x-3)^2}\,dx$$

$$= \lim_{t\to 3^-} \int_0^t (x-3)^{-2}\,dx$$

$$= \lim_{t\to 3^-} -(x-3)^{-1} \Big|_0^t$$

$$= \lim_{t\to 3^-} \frac{-1}{t-3} + \frac{1}{-3}$$

$$= \infty$$

Ex: Evaluate or show that it diverges.

$$\int_0^3 \frac{1}{x-1}\,dx$$

Caution: Asymptote at $x=1$

$$= \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$$

Let $I_1 = \int_0^1 \frac{1}{x-1}\,dx$

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx$$

$$= \lim_{t \to 1^-} \ln|x-1| \Big|_0^t$$

$$= \lim_{t \to 1^-} \ln|t-1| - \ln 1$$

$$= -\infty$$

$I_1$ diverges $\Rightarrow \int_0^3 \frac{1}{x-1}\,dx$ diverges.